

THE ECONOMICS OF INTEREST RATES*

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The paper looks at the behavior of investors in an economy consisting of a production process controlled by a state variable representing the state of technology. The participants in the economy maximize their individual utilities of consumption. Each participant has a constant relative risk aversion. The degrees of risk aversion, as well as the time preference functions, differ across participants. The participants may lend and borrow among themselves, either at a floating short rate, or by issuing or buying term bonds. We derive conditions under which such an economy is in equilibrium, and obtain equations determining interest rates.

1. Introduction

What determines interest rates? Intuitively, it seems that interest rates should be set by supply and demand for borrowing and lending, given the production opportunities in the economy (both current and as they may change in the future depending on technological developments), the time preference for consumption and the attitude toward risk and return of the participants in the economy, and the distribution of wealth across the participants. This would necessitate a general equilibrium model of the economy under the optimal consumption and investment decisions of the players. So far, however, such a model does not seem to have been developed in sufficient generality.

Cox, Ingersoll and Ross (1985a, 1985b) postulate an economy with endogenous production subject to technological changes described by state variables. After identifying the optimal investment and consumption strategies, they derive conditions under which the total riskless lending and the total holdings in debt securities and contingent claims are zero. They then obtain a specific interest rate model under the assumption that the means and variances of the production rates of return are proportional to a single state variable following a square-root process.

This analysis, however, is limited by their assumption that all participants in the economy are identical in their preferences (namely, all with a logarithmic utility function). All investors will thus hold the same portfolio. If there is no borrowing and lending in aggregate, there is no holding of debt securities by any participant. In such an economy, the bond market does not exist. Moreover, since the utility functions are fixed, it does not allow us to study how interest rates depend on the investors' preferences. Dumas (1989) investigates equilibrium conditions in an economy with no technology change and with two investors. Wang (1996) looks at a pure exchange economy with two heterogeneous participants. Chan and Kogan (2002) analyze an exchange economy with heterogeneous

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participants, where each individual's utility is a function of consumption measured in units of an average aggregate endowment.

What is attempted here is an investigation of the term structure of interest rates imposed by equilibrium in a production economy consisting of participants with heterogeneous preferences. When the participants in an economy have different objectives, some will borrow from others in optimizing their investment strategies. Bond prices will be set in such a way that the total demand for borrowing at any maturity equals the total supply. Bond repricing changes the excess returns expected on the production processes and on bonds and thus alters the relative attractiveness of the different investment opportunities. The bond market provides a means of reapportioning the investments in production among the participants in the economy to accommodate their diverse preferences.

We postulate a very simple economy, namely one consisting of a single production process whose behavior is affected by a single variable representing the state of technology. The members of the economy maximize their individual utilities of consumption. It will be assumed that each participant has a constant relative risk aversion. The degrees of risk aversion, as well as the time preference functions, differ across the participants. The participants may lend and borrow among themselves, either at a floating short rate, or by issuing or buying term bonds. In this economy, the total social wealth is invested in the production process and the sum of the bond investments is zero. This provides equilibrium conditions from which we derive equations for the short rate and for the market prices of risk. These relations will allow us to investigate the nature of interest rates. The main difficulty in developing a general equilibrium model with heterogeneous participants, namely, that the aggregate preferences in the economy shift due to changes in the distribution of wealth across the participants, is resolved by showing that the individual wealth levels can be represented as functions of a single process.

We will assume that investment wealth and asset values are measured in terms of a medium of exchange that cannot be stored unless invested in the production process. For instance, this wealth unit may be a perishable consumption good. In this case, interest rates can become negative, because no participant will hold the exchange medium physically but will instead invest it in the production process or lend it to other participants who will put it into production.

2. Optimal investment strategies

Consider an economy consisting of a production process whose rate of return dA/A on an investment A is

$$\frac{dA}{A} = \mu dt + \sigma dy, \quad (1)$$

where $y(t)$ is a Wiener process. The rate of return on an investment in the production process is independent of the investment amount. The development of the production process is affected by a state variable X , $\mu = \mu(X(t), t)$, $\sigma = \sigma(X(t), t)$. The dynamics of the state variable, which can be interpreted as measuring technological change, is given by

$$dX = \zeta dt + \psi dy + \varphi dx, \quad (2)$$

where $x(t)$ is a Wiener process independent of $y(t)$. The parameters ζ , ψ and φ are functions of $X(t)$ and t .

In addition to the production process, the economy allows unrestricted borrowing and lending at any maturity. Denote the interest rate on instantaneous borrowing (the *short rate*) by $r(t)$. An asset $M(t)$ consisting of reinvestment at the short rate,

$$M(s) = M(t) \exp\left(\int_t^s r(\tau) d\tau\right), \quad (3)$$

will be called the money market account.

It will be assumed that it is possible to issue and buy any derivatives of any of the assets and securities in the economy. Specifically, it is possible to short the production process by writing futures against it. It will further be assumed that there are no transaction costs and no taxes or other forms of redistribution of social wealth. We do not explicitly consider firms, since an equity participation in a firm is equivalent to holding a contingent claim on the value of the firm's business.

We will take a shortcut in the development of the equilibrium model. If asset pricing is not free of arbitrage, the economy cannot be in equilibrium. Since there are only two sources of uncertainty, namely the processes y and x , there exist processes λ , η , called the *market prices of risk* for the risk sources y , x , respectively, such that the price P of any asset in the economy must satisfy the equation

$$\frac{dP}{P} = (r + \beta\lambda + \delta\eta) dt + \beta dy + \delta dx, \quad (4)$$

where β , δ are the exposures of the asset to the two risk sources. In particular, we have

$$r = \mu - \sigma\lambda. \quad (5)$$

Alternatively stated, there will exist a numeraire portfolio Z of Long (1990) with the dynamics

$$\frac{dZ}{Z} = (r + \lambda^2 + \eta^2) dt + \lambda dy + \eta dx \quad (6)$$

such that the price P of any asset satisfies

$$P(t) = Z(t) E_t \frac{P(s)}{Z(s)}. \quad (7)$$

Here and throughout, the symbol E_t denotes expectation conditional on a filtration \mathfrak{F}_t generated by $y(t)$, $x(t)$. In integral form, the numeraire portfolio can be written as

$$Z(s) = Z(t) \exp\left(\int_t^s r d\tau + \frac{1}{2} \int_t^s (\lambda^2 + \eta^2) d\tau + \int_t^s \lambda dy + \int_t^s \eta dx\right). \quad (8)$$

The price $B(t,s)$ at time t of a default-free bond with unit face value maturing at time s is given by the equation

$$B(t,s) = E_t \frac{Z(t)}{Z(s)} = E_t \exp\left(-\int_t^s r \, d\tau - \frac{1}{2} \int_t^s (\lambda^2 + \eta^2) \, d\tau - \int_t^s \lambda \, dy - \int_t^s \eta \, dx\right). \quad (9)$$

Term rates will be defined by

$$R(t,\tau) = -\frac{1}{\tau} \log B(t,t+\tau), \quad (10)$$

with $r(t) = R(t,0+)$. Bonds of all maturities, together with the money market account, will be referred to as the bond market. We see from equations (5), (9) and (10) that interest rates are completely described by specifying the market prices of risk $\lambda(t)$ and $\eta(t)$, so our goal is to find out how the two processes are determined in an equilibrium economy.

Suppose that the economy has n participants and let $W_k(0)$ be the initial wealth of the k -th investor. Suppose each investor maximizes the expected utility of lifetime consumption,

$$\max E \int_0^T p_k(t) U_k(c_k(t)) \, dt, \quad (11)$$

where $c_k(t)$ is the rate of consumption at time t , $U_k(c)$ is a utility function with $U_k' > 0$, $U_k'' < 0$, and $p_k(t) \geq 0$, $0 \leq t \leq T$ is a time preference function. We will consider specifically the class of isoelastic utility functions, which we will write in the form

$$\begin{aligned} U_k(c) &= \frac{c^{(\gamma_k-1)/\gamma_k}}{\gamma_k-1} & \gamma_k > 0, \gamma_k \neq 1 \\ &= \log c & \gamma_k = 1. \end{aligned} \quad (12)$$

Here γ_k is the reciprocal of the relative risk aversion coefficient, $1/\gamma_k = -cU_k''/U_k'$. We will call γ_k the *risk tolerance*.

An investment strategy is fully described by the exposures $\beta_k(t)$ and $\delta_k(t)$ to the sources of risk y and x . The wealth $W_k(t)$ at time t grows by the increment

$$dW_k = W_k(r + \beta_k \lambda + \delta_k \eta) \, dt + W_k \beta_k \, dy + W_k \delta_k \, dx - c_k \, dt. \quad (13)$$

Let $V_k(t)$ be the value at time t of the expected utility of consumption under an optimal investment and consumption strategy,

$$V_k(t) = \max E_t \int_t^T p_k(s) U_k(c_k(s)) \, ds. \quad (14)$$

Under some mild regularity conditions (cf. Fleming and Rishel, 1975), a necessary and sufficient condition for optimality is given by the Bellman equation

$$\max (E \, dV_k + p_k U_k(c_k) \, dt) = 0. \quad (15)$$

Put

$$\begin{aligned}
V_k &= \frac{1}{\gamma_k - 1} Q_k^{1/\gamma_k} W_k^{(\gamma_k - 1)/\gamma_k} & \gamma_k > 0, \gamma_k \neq 1 \\
&= Q_k \log W_k + G_k & \gamma_k = 1,
\end{aligned} \tag{16}$$

with the dynamics of Q_k written as

$$\frac{dQ_k}{Q_k} = \vartheta_k dt + \theta_k dy + \omega_k dx. \tag{17}$$

Calculating EdV_k yields the equation

$$\begin{aligned}
\max\left(\frac{1}{\gamma_k(\gamma_k - 1)} \vartheta_k + \frac{1}{\gamma_k} (r + \beta_k \lambda + \delta_k \eta - \frac{c_k}{W_k})\right. \\
\left. - \frac{1}{2} \frac{1}{\gamma_k^2} ((\beta_k - \theta_k)^2 + (\delta_k - \omega_k)^2)\right) Q_k^{1/\gamma_k} W_k^{(\gamma_k - 1)/\gamma_k} + p_k U_k(c_k) = 0.
\end{aligned} \tag{18}$$

Maximization over the values of β_k , δ_k and c_k yields a unique maximum attained at the point

$$\beta_k = \gamma_k \lambda + \theta_k \tag{19}$$

$$\delta_k = \gamma_k \eta + \omega_k \tag{20}$$

$$c_k = \frac{p_k^{\gamma_k} W_k}{Q_k}. \tag{21}$$

The investment position of each participant is independent of his current wealth level W_k and the rate of consumption is proportional to the current wealth.

Substituting these values back into (18), we get the equation

$$\vartheta_k + (\gamma_k - 1)(r + \lambda \theta_k + \eta \omega_k) + \frac{1}{2} \gamma_k (\gamma_k - 1)(\lambda^2 + \eta^2) + \frac{p_k^{\gamma_k}}{Q_k} = 0 \tag{22}$$

and consequently

$$\frac{dQ_k}{Q_k} = \left(-(\gamma_k - 1)(r + \lambda \theta_k + \eta \omega_k) - \frac{1}{2} \gamma_k (\gamma_k - 1)(\lambda^2 + \eta^2) - \frac{p_k^{\gamma_k}}{Q_k}\right) dt + \theta_k dy + \omega_k dx. \tag{23}$$

We note that

$$Ed(Q_k Z^{\gamma_k - 1}) = -p_k^{\gamma_k} Z^{\gamma_k - 1} dt \tag{24}$$

and integrating subject to the condition

$$Q_k(T) = 0 \tag{25}$$

we get

$$Q_k(t) = Z^{1-\gamma_k}(t) E_t \int_t^T p_k^{\gamma_k}(\tau) Z^{\gamma_k-1}(\tau) d\tau. \quad (26)$$

The wealth increment can be determined as

$$\frac{dW_k}{W_k} = (r + \lambda\theta_k + \eta\omega_k + \gamma_k(\lambda^2 + \eta^2)) dt + (\gamma_k\lambda + \theta_k) dy + (\gamma_k\eta + \omega_k) dx - \frac{P_k^{\gamma_k}}{Q_k} dt. \quad (27)$$

Comparing equations (6), (23) and (27), we find that

$$d\left(\frac{W_k}{Q_k} Z^{-\gamma_k}\right) = 0. \quad (28)$$

On integration,

$$W_k(t) = v_k Z^{\gamma_k}(t) Q_k(t) \quad (29)$$

and therefore

$$W_k(t) = v_k Z(t) E_t \int_t^T p_k^{\gamma_k}(\tau) Z^{\gamma_k-1}(\tau) d\tau, \quad (30)$$

where

$$v_k = \frac{W_k(0)}{Q_k(0)} Z^{-\gamma_k}(0) = \frac{W_k(0)}{Z(0) E \int_0^T p_k^{\gamma_k}(\tau) Z^{\gamma_k-1}(\tau) d\tau} \quad (31)$$

is a constant. The behavior of the individual wealth levels W_k is fully determined by the process Z .

The optimal rate of consumption is

$$c_k = v_k p_k^{\gamma_k} Z^{\gamma_k}. \quad (32)$$

We see from equation (30) that, when measured in units of the numeraire portfolio, the current wealth is equal to the expected total future consumption.

3. The equilibrium economy

If we consider the economy as a whole, the total wealth must be invested in the production process. Any lending and borrowing is among the participants in the economy, and its sum must be zero. Thus, the total exposure to the process y is that of the total wealth invested in the production, and the total exposure to the process x is zero. The conditions for equilibrium are then

$$\sum_{k=1}^n \beta_k W_k = \sigma W \quad (33)$$

$$\sum_{k=1}^n \delta_k W_k = 0, \quad (34)$$

where

$$W = \sum_{k=1}^n W_k \quad (35)$$

is the total social wealth.

Using the relation (29) and substituting back from (19) and (20), write equation (27) as

$$dW_k = W_k(r + \beta_k \lambda + \delta_k \eta) dt + W_k \beta_k dy + W_k \delta_k dx - v_k p_k^{\gamma_k} Z^{\gamma_k} dt \quad (36)$$

and sum over all investors. This produces the equation

$$dW = \mu W dt + \sigma W dy - \sum_{k=1}^n v_k p_k^{\gamma_k} Z^{\gamma_k} dt \quad (37)$$

describing the dynamics of the total wealth. The first two terms on the right-hand side correspond to the investment of the total social wealth in the production, and the third term represents the total consumption. The terminal condition is $W(T) = 0$.

The unique solution of the stochastic differential equation (37) is given by

$$W(t) = Z(t) E_t \int_t^T \sum_{k=1}^n v_k p_k^{\gamma_k}(\tau) Z^{\gamma_k-1}(\tau) d\tau. \quad (38)$$

Indeed, we can write (37) as

$$d\left(\frac{W}{A}\right) = -\frac{1}{A} \sum_{k=1}^n v_k p_k^{\gamma_k} Z^{\gamma_k} dt \quad (39)$$

and therefore

$$E d\left(\frac{W}{Z}\right) = E d\left(\frac{W}{A} \frac{A}{Z}\right) = \frac{A}{Z} d\left(\frac{W}{A}\right) + \frac{W}{A} E d\left(\frac{A}{Z}\right) = -\sum_{k=1}^n v_k p_k^{\gamma_k} Z^{\gamma_k-1} dt, \quad (40)$$

due to the property (7) of the numeraire process. Equation (38) follows by integration.

To determine the process Z , however, we need the solution of (37) in a more explicit form. We see from equation (37) that the only state variable for W besides X is the value of Z . Write $W(t) = W(X, Z, t)$ as a function of the state variables. Then

$$dW = W_t dt + (\zeta dt + \psi dy + \phi dx) W_x + ((\mu - \sigma\lambda + \lambda^2 + \eta^2) dt + \lambda dy + \eta dx) Z W_z + \frac{1}{2}(\psi^2 + \phi^2) W_{xx} dt + (\psi\lambda + \phi\eta) Z W_{xz} dt + \frac{1}{2}(\lambda^2 + \eta^2) Z^2 W_{zz} dt, \quad (41)$$

where the subscripts X , Z and t denote partial derivatives with respect to these variables. Comparing equations (37) and (41), we must have

$$\psi W_x + \lambda Z W_z = \sigma W \quad (42)$$

$$\phi W_X + \eta Z W_Z = 0. \quad (43)$$

Solving for λ , η and substituting produces the equation

$$W_t = -\mathfrak{R}[W, t] - \sum_{k=1}^n v_k p_k^{\gamma_k} Z^{\gamma_k}, \quad (44)$$

where

$$\begin{aligned} \mathfrak{R}[W, t] = & (\zeta + \sigma\psi)W_X + \mu Z W_Z + \frac{1}{2}(\psi^2 + \phi^2)W_{XX} + (\sigma\psi W - (\psi^2 + \phi^2)W_X) \frac{W_{XZ}}{W_Z} \\ & + ((\psi^2 + \phi^2)W_X^2 - 2\sigma\psi W W_X + \sigma^2 W^2) \left(\frac{1}{Z W_Z} + \frac{1}{2} \frac{W_{ZZ}}{W_Z^2} \right) - (\mu + \sigma^2)W \end{aligned} \quad (45)$$

is an operator that involves only derivatives with respect to X and Z . Equation (44) is subject to the condition

$$W(T) = 0. \quad (46)$$

The value of $\mathfrak{R}[W, T]$ is defined by its limit for $t \rightarrow T$. From (38), we have

$$W(t) \square \sum_{k=1}^n v_k Z^{\gamma_k}(t) \int_t^T p_k^{\gamma_k}(\tau) d\tau, \quad t \rightarrow T. \quad (47)$$

If none of the time preference functions $p_k(t)$ has an atom at T , the limit is $\mathfrak{R}[W, T] = 0$. This assumes that the sum of the integrals in (47) is nonzero for all $t < T$, in other words, that at least one participant in the economy assigns positive utility to consumption up to the date T . If it is zero for $T_1 < t < T$ but positive for all $t < T_1$, the boundary conditions are applied to T_1 .

Once the function $W(X, Z, t)$ has been determined, λ and η are calculated as

$$\lambda = \frac{\sigma W - \psi W_X}{Z W_Z} \quad (48)$$

$$\eta = -\frac{\phi W_X}{Z W_Z}. \quad (49)$$

To demonstrate that the process W is indeed a function of X , Z and t only, assume to the contrary that there are other state variables (for instance, the current and past values of the individual wealth levels W_k) of which W is a function. Suppose Y (possibly a vector) is such a variable, $W(t) = W(X, Y, Z, t)$. In that case, the market prices of risk $\lambda = \lambda(X, Y, Z, t)$, $\eta = \eta(X, Y, Z, t)$ are functions of Y as well and the dynamics of Z depends on Y . Write the dynamics of Y as $dY = \chi_0 dt + \chi_1 dy + \chi_2 dx$, where $\chi_i = \chi_i(X, Y, Z, t)$, $i = 0, 1, 2$. Expressing dW by Ito's lemma and comparing the coefficients of dt , dy and dx with those of (37), we can again eliminate λ , η and obtain a partial differential equation in X , Y , Z and t . But the only coefficients in that equation that depend on Y are the χ_i , all of which are multiplied by derivatives with respect to Y . Therefore, any solution of (44) is also a solution of that

equation. Since W is unique, it must be independent of Y . Consequently, λ and η are functions of X , Z and t only. The process $(X(t), Z(t))$ is Markov.

Equations (44), (48) and (49) define λ and η , and the process Z is given by its dynamics (6). Bond prices and rates are then determined by (5), (9) and (10). This constitutes a complete solution of the problem.

Equation (44) is an evolution equation. Very little is known about nonlinear partial differential equations in general, and the equation needs to be investigated case by case. For some of the problems that may be encountered in the presence of nonlinearity see, for instance, Li and Chen (1992) or Logan (1994). We can expect, however, that the reasons for ill behavior of the solution will often be an economic misspecification rather than mathematical irregularity. For instance, if $\mu(X,t)$ is too steep a function of the state variable X and X is allowed to drift to large values too freely (as in example 7 below), the production process $A(t)$ may explode or have an infinite expectation. The process Z will not exist and equation (44) will have no solution.

In well-posed situations, equation (44) is easy to solve computationally. The simplest method is to replace the derivative W_t by the difference quotient and recursively calculate $W(t-h)$ from $W(t)$. For some guidance on numerical methods see, for instance, Ganzha and Vorozhtsov (1996). The main computational difficulty is the necessity to iterate on the values of the constants v_1, v_2, \dots, v_n , since they are determined (up to a scalar) by (31) only after Z has been found.

4. Examples

Example 1. Suppose $\gamma_k = \gamma$, $k = 1, 2, \dots, n$ (although the investors may still differ by their time preference functions $p_k(t)$). Then the solution of equation (44) is

$$W = Z^\gamma F, \quad (50)$$

where $F = F(X,t)$ is the solution of

$$\begin{aligned} F_t + \left(\zeta + \frac{\gamma-1}{\gamma} \sigma \psi\right) F_x + \frac{1}{2} (\psi^2 + \phi^2) F_{xx} - \frac{\gamma-1}{2\gamma} (\psi^2 + \phi^2) \frac{F_x^2}{F} \\ + (\gamma-1) \left(\mu - \frac{1}{2\gamma} \sigma^2\right) F + \sum_{k=1}^n v_k p_k^\gamma = 0 \end{aligned} \quad (51)$$

subject to $F(X,T) = 0$. The process Z is given by

$$Z(t) = W^{1/\gamma}(0) A^{-1/\gamma}(0) A^{1/\gamma}(t) F^{-1/\gamma}(X(t), t) \exp\left(-\frac{1}{\gamma} \int_0^t \frac{\sum v_k p_k^\gamma(\tau)}{F(X(\tau), \tau)} d\tau\right).$$

The constants v_1, v_2, \dots, v_n are determined by the equations

$$W_k(0) = v_k W(0) A^{(1-\gamma)/\gamma}(0) F^{-1/\gamma}(X(0), 0) \\ \cdot \mathbb{E} \int_0^T p_k^\gamma(t) A^{(\gamma-1)/\gamma}(t) F^{(1-\gamma)/\gamma}(X(t), t) \exp\left(\frac{1-\gamma}{\gamma} \int_0^t \frac{\sum v_k p_k^\gamma(\tau)}{F(X(\tau), \tau)} d\tau\right) dt.$$

Then

$$\lambda = \frac{1}{\gamma} \left(\sigma - \psi \frac{F_X}{F} \right)$$

$$\eta = -\frac{\varphi F_X}{\gamma F}.$$

The dynamics of $\lambda = \lambda(X, t)$ and $\eta = \eta(X, t)$, as well as that of the short rate $r = \mu - \sigma\lambda$, follow from the dynamics of X .

Example 2. If, in particular, $\gamma_k = 1$, $k = 1, 2, \dots, n$, then

$$F(X, t) = \sum_{k=1}^n v_k \int_t^T p_k(\tau) d\tau$$

and

$$Z(t) = \frac{W(0)}{A(0)F(X(0), 0)} A(t).$$

Solving (31) for v_1, v_2, \dots, v_n , we get

$$v_k = \frac{N W_k(0)}{\int_0^T p_k(\tau) d\tau},$$

where N is an arbitrary multiplier. On substitution, we have

$$F(X, t) = N \sum_{k=1}^n W_k(0) \int_t^T p_k(\tau) d\tau \bigg/ \int_0^T p_k(\tau) d\tau.$$

The prices of risk are

$$\lambda = \sigma$$

$$\eta = 0$$

and the short rate is

$$r = \mu - \sigma^2.$$

Example 3. Let $\gamma_k = \gamma$, $k = 1, 2, \dots, n$ and suppose there are no unforeseen technological changes, so that μ and σ are functions of time only. Then F is a function of t only and we have $\lambda = \sigma/\gamma$, $\eta = 0$, and

$$r = \mu - \frac{1}{\gamma} \sigma^2.$$

Interest rates are deterministic, independent of the time preference functions $p_k(t)$.

Example 4. Suppose that the time preference functions of all participants are concentrated at the point T . In other words, each participant maximizes the expected utility of end-of-period wealth. Then

$$\begin{aligned} W(t) &= \frac{W(0)}{A(0)} A(t) & t < T \\ &= 0 & t = T. \end{aligned}$$

At T , we have

$$W(T-) = \sum_{k=1}^n v_k Z^{\gamma_k}(T).$$

Put

$$K(Z) = \frac{A(0)}{W(0)} \sum_{k=1}^n v_k Z^{\gamma_k}$$

and denote by K^{-1} the inverse function of K . Since

$$\frac{A(t)}{Z(t)} = E_t \frac{A(T)}{Z(T)},$$

we get

$$Z(t) = \frac{A(t)}{E_t \frac{A(T)}{K^{-1}(A(T))}}$$

and bond prices are given by

$$B(t, s) = \frac{A(t)}{E_t \frac{A(T)}{K^{-1}(A(T))}} E_t \frac{A(T)}{A(s) K^{-1}(A(T))}.$$

Example 5. Suppose that the time preference functions of all participants are concentrated at the point T , and assume moreover that $\gamma_k = \gamma$, $k = 1, 2, \dots, n$ (so that investors have homogeneous preferences). Then

$$K^{-1}(A) = NA^{1/\gamma},$$

where N is a constant multiplier, and

$$Z(t) = \frac{NA(t)}{E_t A^{(\gamma-1)/\gamma}(T)}. \quad (52)$$

For instance, suppose that

$$\begin{aligned} \mu &= X \\ \zeta &= \alpha(\bar{X} - X) \end{aligned}$$

and let σ , ψ and ϕ be constant. Evaluating the expectation in (52) gives

$$Z(t) = A^{1/\gamma}(t) \exp\left(\frac{1-\gamma}{\gamma} D(t, T)X + g(t)\right),$$

where $g(t)$ is a function of t alone, and

$$D(t, T) = \frac{1}{\alpha}(1 - e^{-\alpha(T-t)}). \quad (53)$$

Alternatively, we can solve equation (51) and find F in the form

$$F(X, t) = \exp((\gamma-1)D(t, T)X + h(t)). \quad (54)$$

Consequently, we have

$$\lambda = \frac{\sigma}{\gamma} + \frac{1-\gamma}{\gamma} \psi D \quad (55)$$

$$\eta = \frac{1-\gamma}{\gamma} \phi D \quad (56)$$

$$r = X - \frac{\sigma^2}{\gamma} + \frac{\gamma-1}{\gamma} \sigma \psi D. \quad (57)$$

The dynamics of r is given by

$$dr = \alpha(\bar{r} - r)dt + \psi dy + \phi dx,$$

where \bar{r} is a function of time. Interest rates of all maturities are Gaussian, and market prices of both sources of risk are functions of time only. All investors hold the same portfolio, $\beta_k = \sigma$, $\delta_k = 0$.

Example 6. Make the same assumptions as in example 5, but let

$$\begin{aligned}\sigma^2 &= \hat{\sigma}^2 X \\ \psi^2 &= \hat{\psi}^2 X \\ \varphi^2 &= \hat{\varphi}^2 X,\end{aligned}$$

where $\hat{\sigma}$, $\hat{\psi}$ and $\hat{\varphi}$ depend on time only. The function F still has the form (54), with D given by a different expression than (53) but still independent of X . Equations (55), (56) and (57) hold, and we have

$$\begin{aligned}\lambda &= \hat{\lambda}\sqrt{X} \\ \eta &= \hat{\eta}\sqrt{X} \\ r &= \zeta X\end{aligned}$$

with $\hat{\lambda}$, $\hat{\eta}$ and ζ being functions of time only. The dynamics of r is described by

$$dr = \kappa(\bar{r} - r)dt + \sqrt{r}(\xi_1 dx + \xi_2 dy),$$

where κ , \bar{r} , ξ_1 , ξ_2 are functions of time. This is a model of the Cox, Ingersoll, Ross type.

Example 7. Consider the same situation as in the previous two examples, but let

$$\begin{aligned}\sigma &= \hat{\sigma}X \\ \psi &= \hat{\psi}X \\ \varphi &= \hat{\varphi}X\end{aligned}$$

with $\hat{\sigma}$, $\hat{\psi}$ and $\hat{\varphi}$ constant. If $\gamma > 1$, the expectation in (52) is infinite. We have $Z(t) \equiv 0$ and the numeraire portfolio does not exist. Equilibrium cannot be attained in this economy.

5. Term structure models

A number of specific models of the term structure of interest rates have been proposed, derived from the principle of no arbitrage. We wish to ask the following question: For a given term structure model, does an equilibrium economy of the kind investigated here exist in which interest rates are governed by that model?

If an equilibrium exists, the expectations in equations (30) must be finite. For that, it is necessary that

$$E_t Z^{\gamma_k - 1}(s) < \infty \quad (58)$$

for all $k = 1, 2, \dots, n$, $0 \leq t \leq s \leq T$. On the other hand, if (58) holds, it is always possible to construct an economy in equilibrium (cf. Harrison and Kreps, 1979). We will therefore investigate whether condition (58) is satisfied by a given term structure model.

We will look specifically at one-factor interest rate models. We obtain such models in the economy proposed here if there is only one source of risk. This will happen if, for instance, $\sigma = 0$, $\psi = 0$. These models then have the form

$$B(t, s) = E_t \frac{Z(t)}{Z(s)} = E_t \exp\left(-\int_t^s r \, d\tau - \frac{1}{2} \int_t^s \eta^2 \, d\tau - \int_t^s \eta \, dx\right).$$

This question was investigated in some detail in Vasicek (2000), who gives a somewhat different rationale for the condition (58). It is shown that a Gaussian model always satisfies the finiteness condition. On the other hand, consider the Cox, Ingersoll, Ross model described by

$$dr = \alpha(\bar{r} - r)dt + \xi\sqrt{r} \, dx$$

$$\eta = \hat{\eta}\sqrt{r}.$$

Note that $\xi\hat{\eta}$ is negative when the bond risk premia $E dB/B - rdt = \eta E dx dB/B$ are positive. Put

$$a = \gamma_{\max} \alpha^2 + (1 - \gamma_{\max})((\alpha + \xi\hat{\eta})^2 + 2\xi^2)$$

$$b = \alpha + (1 - \gamma_{\max})\xi\hat{\eta},$$

where γ_{\max} is the largest of γ_k , $k = 1, 2, \dots, n$. The expectation in (58) is finite if and only if $a \geq 0$, or $a < 0$ and

$$s < t + \frac{2}{\sqrt{-a}} (\pi - \arctan(\sqrt{-a}/b)).$$

When applying a term structure model (for instance, in derivatives pricing), one does not want to make assumptions about the preferences of the participants in the economy that generated that model. In other words, it is desirable to know under what conditions the model is consistent with an equilibrium economy with *any* participant preferences. For the Cox, Ingersoll, Ross model, in order that (58) holds for all γ_k and all $t \leq s$, it is necessary (and sufficient) that

$$(\alpha + \xi\hat{\eta})^2 + 2\xi^2 \leq \alpha^2. \quad (59)$$

The inequality (59) (which can be written as $\gamma \leq \kappa$ in the notation of their 1985b paper) is a restriction on the parameters of the model in order that it may describe the behavior of interest rates in an economy with arbitrary preferences of the participants.

For the Black, Derman, Toy (1990) model, the expectation in (58) is infinite for all $\gamma_k > 1$ and $s > t$. No equilibrium economy exists in which the bond market follows this model. There would be an infinite demand for interest rate swaps (receiving floating and paying fixed rates) with no supply.

6. Conclusions

This paper looks at the behavior of heterogeneous investors in an economy consisting of a production process and a bond market. If each participant in the economy pursues a strategy optimal with respect to his preferences, the market has to accommodate the resultant demand and supply of credit by pricing risk so that the economy stays in equilibrium. We derive conditions under which such equilibrium is possible, and obtain equations determining interest rates. These results can be used for quantitative analyses of various economic phenomena.

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