Portfolio Choice with Capital Gain Taxation and the Limited Use of Losses*

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Current Draft: February 2014

*We thank Wolfgang Buehler, Victor DeMiguel, Pascal François, Lorenzo Garlappi, Bruce Grundy, Christian Heyerdahl-Larsen, Philipp Illeditsch, Urban Jermann, Spencer Martin, Jeffery Pontiff, Neal Stoughton, Neng Wang, Alan White, Avi Wohl, Amir Yaron, Harold Zhang, and seminar participants at Australia National University, the University of Melbourne, the University of New South Wales, and the Wharton School of the University of Pennsylvania. An earlier version of the paper that only incorporated a single stock analysis was presented, among other places, at the Western Finance Association meeting, the European Finance Association Meeting, and the UBC Summer Conference. We acknowledge funding support from the Research Council of Norway (Ehling), the Centre for Asset Pricing Research (CAPR) at BI (Ehling and Yang), and the DeMong-Pettit Research Fund at the McIntire School of Commerce (Gallmeyer). We also thank the Texas Advanced Computing Center for providing computing resources.

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Abstract

Portfolio Choice with Capital Gain Taxation and the Limited Use of Losses

We study portfolio choice with multiple stocks and capital gain taxation assuming that capital losses can only be used to offset current or future realized capital gains. We show both numerically and through a backtesting analysis that average optimal equity holdings are significantly lower compared to the case typically studied in the literature where the use of capital losses is unrestricted. With low diversification benefits or high embedded capital gains, allocations remain under-diversified even if embedded capital losses are sufficient to offset the gains.

Keywords: portfolio choice, capital gain taxation, limited use of capital losses

JEL Classification: G11, H20

1 Introduction

Typically, studies of portfolio choice with capital gains taxation focus on two effects: a *capitalization effect* where the capital gain tax lowers the allocation to equity and a *lock-in effect* where embedded capital gains reduce the willingness to sell. We show that capital gains taxation with the *limited use of losses* (LUL) significantly increases the sensitivity of the optimal trading strategy to the capitalization effect relative to the lock-in effect.[1] When losses can only be used to offset current or future realized capital gains, an investor's equity holdings are significantly lower and exhibit strong time-variation across up and down equity markets compared to the case where the use of capital losses is unrestricted.

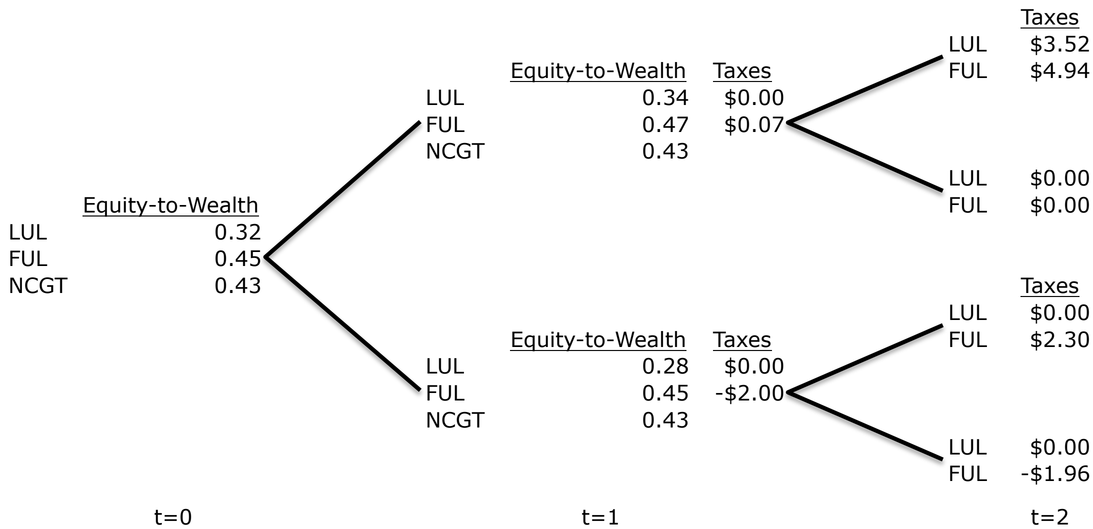
In the extant academic literature, however, it is commonly assumed that the use of capital losses is unrestricted, termed here the *full use of losses* (FUL): if capital losses are larger than capital gains in a period, the investor receives a tax rebate that cushions the downside of holding equity. While we would expect tax rebates to boost the demand for equity relative to the LUL case, what is surprising is the magnitude of the difference. For example, we document that an FUL investor holds even more equity than an untaxed investor when the portfolio contains no capital gains.

We illustrate the impact of alternative capital gain tax assumptions in a simple portfolio choice problem with one stock and a bond. Consider binomial uncertainty for the stock, two trading dates, and a final date where the portfolio is liquidated. The investor maximizes after-tax final period wealth with constant relative risk aversion utility and an initial endowment of $100 with no embedded capital gains or losses. Figure 1 summarizes optimal portfolio choice expressed as an equity-to-wealth ratio and capital gain taxes paid through the binomial tree under both LUL- and FUL-based capital gain tax systems as well as the no capital gain tax benchmark denoted NCGT. Up (down) moves in the binomial tree denote stock price increases (decreases). Additional details are provided in Section 3.

The initial equity-to-wealth ratio provides a concise ex-ante measure of the capitalization effect when compared to the NCGT benchmark since it directly measures the change in the allocation to equity in the presence of taxes. From Figure 1, the LUL investor initially trades to an equity-to-wealth ratio of 0.32, which is significantly below the constant equity-to-wealth ratio of 0.43 under the NCGT

[1] Our work is motivated by Gallmeyer and Srivastava (2011) who study no arbitrage restrictions on after-tax price systems with the limited use of capital losses. To our knowledge, this was the first work that explored the limited use of capital losses in capital gain tax problems. See Marekwica (2012) for other recent work on the role of the limited use of capital losses in portfolio choice problems when there are asymmetries between the tax treatment of profits and losses. Further, for early related work see Domar and Musgrave (1944) who explore the role of losses on risk sharing when taxes are assessed on excess returns and Stiglitz (1969) who studies the impact of losses on portfolio choice with income taxes.

Figure 1: **Motivating Example**



benchmark strategy. The FUL investor's initial equity-to-wealth ratio, at 0.45, is higher than the NCGT benchmark. From an after-tax risk-return tradeoff perspective, an allocation above the NCGT benchmark is possible. If the tax reduces the volatility of after-tax returns more than the after-tax risk premium, the after-tax Sharpe ratio is pushed higher implying a higher demand for equity than the NCGT investor. However, the LUL case, which accounts for capital losses in a way which is consistent with all tax codes we are aware of, shows that this intuition is misleading and that the FUL case grossly underestimates the capitalization effect. The FUL investor's increased equity demand is driven by the prospect of artificially cushioning the impact of a stock price drop through a tax rebate. If the stock price drops at $t = 1$, a tax rebate of $2 is collected which immediately increases the FUL investor's wealth.

The impact of the lock-in effect is captured by examining the optimal trade when the investor is overexposed to equity with embedded capital gains. If the stock price increases at $t = 1$, both the LUL and the FUL investors hold equity with embedded capital gains. Given the LUL investor started with a smaller investment, the equity-to-wealth ratio of 0.34 is again smaller than the NCGT benchmark. The FUL investor still holds the most equity with an equity-to-wealth ratio of 0.47.

Comparing across the $t = 1$ up and down stock price paths, the LUL investor's equity-to-wealth ratio varies the most over time as the capitalization effect drives equity holdings lower in the down stock price path. In contrast, the prospect of generating an additional tax rebate at $t = 2$ of $1.96 still keeps the FUL investor's position elevated at $t = 1$ in the down stock price path. This simple example highlights the interplay between how capital losses are treated and how optimal equity holdings time-vary as the relative tradeoffs of the capitalization and lock-in effects change.

To assess the impact of the limited use of losses, we solve a long-horizon portfolio choice problem with security price dynamics chosen to be largely consistent with empirical moments of U.S. large-capitalization stock indexes. We use tax rate parameters consistent with the U.S. tax code as well as the tax codes in many European countries and Canada. Specifically, we modify the single stock model of Dammon, Spatt, and Zhang (2001b) and the multiple stock model of Gallmeyer, Kaniel, and Tompaidis (2006) by restricting the use of losses. We use a test region iterative contraction method, Yang and Tompaidis (2013), to solve our optimization problem with multiple stocks since it allows us to handle several endogenous state variables that are due to capital gain taxation and the limited use of losses.[2]

Our long-horizon asset allocation problem illustrates that the limited use of losses reduces the investors' allocation to equity and induces strong time-variation in allocations across up and down markets. Comparative static analysis unambiguously supports the robustness of our main result.

To get a first insight into the impact of portfolio choice with multiple stocks and capital gain taxation assuming that capital losses can only be used to offset current or future realized capital gains, we examine how two-stock strategies behave when the investor is overinvested in equity and simultaneously has an embedded capital gain and loss in the portfolio. For this case, it is optimal to tax-loss sell the position with the embedded capital loss and reduce the position in the stock with embedded capital gains, using realized losses to offset the gains. The cheaper it is to reduce the overinvestment in the equity in the lock-in region, i.e., the closer the stock's relative basis is to one, the larger is the after-trade allocation in the stock with the embedded capital loss to optimize the total equity allocation. When diversification benefits are relatively low or capital gains taxes are relatively high, allocations can remain under-diversified even after all embedded capital losses have been used to offset embedded capital gains. Overall, we find that the differences between the FUL and LUL conditional trading strategies remain large in the case of multiple stocks.

The constrained realism of the portfolio choice problem studied still overstates the capital gain lock-in effect, especially over the lifetime of an investor. Introducing liquidation shocks into our

[2]While alternative numerical solution approaches, such as the one in Brandt et al. (2005), Gallmeyer, Kaniel, and Tompaidis (2006), and Garlappi and Skoulakis (2008), exist, due to the dimensionality and singularity in the dynamics of our problem, we were unable to implement them.

framework shows that the economic costs between the LUL and FUL investors increase from slightly above two and a half percent to six percent of the lifetime wealth equivalent of an untaxed investor. Additionally, using historical return data, we show that several decades may pass by before the LUL equity holdings converge to the FUL equity holdings. With liquidation shocks, we find that optimal equity holdings of the LUL investor remain below the holdings of the FUL investor almost over the entire lifetime of the investors. On average, we find that the backtested LUL equity-to-wealth ratio is below the FUL ratio by 10% at age 20 and remains below the FUL allocation by, at least, 3% until the age of 50. Overall, the theoretical and empirical examples suggest that the difference between the LUL and FUL strategies is large and economically significant.

2 The Consumption-Portfolio Problem

The investor chooses an optimal consumption and investment policy in the presence of realized capital gain taxation at trading dates $t = 0, ..., T$. Our assumptions concerning the exogenous price system, taxation, and the investor's portfolio problem are outlined below. The notation and model structure are based on Gallmeyer, Kaniel, and Tompaidis (2006) where we modify capital gain taxation to accommodate for the limited use of capital losses. A full description of our partial equilibrium setting is given in Appendix A.

2.1 Security Market

The set of financial assets available to the investor consists of a riskless money market and multiple dividend-paying stocks. Stocks pay dividends with constant dividend yields and ex-dividend stock price processes evolve with independent increments given by lognormal distributions. The money market pays a continuously-compounded pre-tax rate of return r.

2.2 Taxation

Dividends and interest income are taxed as ordinary income on the date they are paid at rates τ_D and τ_I, respectively.

Our analysis centers around a feature of the tax code that has received little attention in the literature, namely that most capital gain tax codes restrict how realized capital losses are used. The most common assumption used in the portfolio choice literature is that there are no restrictions on the use of capital losses, which we term the *full use of capital losses* case.

Definition 1. *Under the full use of capital losses (FUL) case, an investor faces no restrictions on the use of realized capital losses. When realized capital losses are larger than realized capital gains in a period, the remaining capital losses generate a tax rebate that can be immediately invested.*

Definition 1 is assumed in several papers that study portfolio choice with capital gain taxes (Constantinides (1983); Dammon, Spatt, and Zhang (2001a,b, 2004); Garlappi, Naik, and Slive (2001); Hur (2001); DeMiguel and Uppal (2005); Gallmeyer, Kaniel, and Tompaidis (2006)).[3]

Given most tax codes restrict the use of capital losses, our alternative form of realized capital gain taxation is referred to as the *limited use of capital losses* case.

Definition 2. *Under the limited use of capital losses (LUL) case, an investor can only use realized capital losses to offset current realized capital gains. Unused capital losses can be carried forward indefinitely to future trading dates.*

The no-arbitrage analysis in Gallmeyer and Srivastava (2011) shows that, under the LUL case, an investor is indifferent between realizing an unused capital loss or carrying it forward. Based on this result, we assume that, for the FUL and LUL case, the investor immediately realizes all capital losses each period even if they are not used.

Our definition of the limited use of capital losses does not include the ability to use capital losses to offset current taxable income.[4] Additionally, our analysis does not distinguish between differential taxation of long and short-term capital gains since our investors trade annually.[5]

Realized capital gains and losses are subject to a constant capital gain tax rate τ_C. The tax basis used for computing realized capital gains or losses is calculated as a weighted-average purchase price.[6]

[3]The FUL assumption has the computational advantage that capital losses are never carried over and hence the investor does not need to keep track of an additional state variable.

[4]In the U.S. tax code, individual investors can only offset up to $3,000 of taxable income per year with realized capital losses. Allowing for this tax provision requires keeping track of wealth as an extra state variable. Marekwica (2012) shows that asymmetries in the tax code such as the $3,000 dollar rule introduce the incentive to periodically realize capital gains to allow for using realized losses in the future for tax rebates on income. This feature of the U.S. tax code favors poor LUL investors but likely has only a small impact on most investors. Further, the relevance of the $3,000 dollar rule has decreased considerably over time as the capital loss limit has not increased since 1978.

[5]For such an analysis, see Dammon and Spatt (1996).

[6]The U.S. tax code allows for a choice between weighted-average price and exact identification of the shares to be sold. The Canadian and European tax codes use the weighted-average price rule. While choosing to sell the shares with the smallest embedded gains using the exact identification rule is beneficial to the investor, solving for the optimal investment strategy becomes numerically intractable for a large number of trading periods given the dimension of the state variable

When an investor dies, capital gain taxes are forgiven and tax bases of stocks owned reset to current market prices. This is consistent with the reset provision in the U.S. tax code. Dividend and interest taxes are still paid at the time of death. We also consider the case when capital gain taxes are not forgiven which is consistent with the Canadian and many European tax codes. While we allow investors to "wash sell" and immediately rebalance after they realize capital losses, they are precluded from shorting the stock which eliminates a "shorting against the box" transaction to avoid paying capital gain taxes.[7]

2.3 Investor Problem

To finance consumption, the investor trades in risky stocks and the money market. The setting we have in mind is one where a taxable investor trades individual stocks or exchange traded funds (ETFs).[8] Given an initial equity endowment, a consumption and security trading policy is *admissible* if it is self-financing, involves no short selling of stocks, and leads to nonnegative wealth over the investor's lifetime. The investor lives at most T periods and faces a positive probability of death each period.

Over his lifetime, there may be times when the investor is forced to liquidate either a stock or his entire portfolio or consume an extraordinary amount of his wealth. We consider such an extension following Constantinides (1983). We assume that when forced liquidation occurs, the investor liquidates at least a fraction, F, of his wealth. Forced liquidation follows a Poisson process with arrival rate Λ. With multiple stocks, the asset with the lowest tax cost is liquidated first, i.e., the stock with the highest relative basis, followed by the asset with the next lowest tax cost. A full description of how we introduce liquidation into the model is given in Appendix A.

The investor's objective is to maximize expected utility of real lifetime consumption and a time of death bequest motive by choosing an admissible consumption-trading strategy given an initial

increases with time (Dybvig and Koo, 1996; Hur, 2001; DeMiguel and Uppal, 2005). However, for parameterizations similar to those in this paper, DeMiguel and Uppal (2005) numerically show that the certainty-equivalent wealth loss using the weighted-average price basis rule as compared to the exact identification rule is small.

[7] A wash sale is a sale of a financial security with an embedded capital loss and a proximate repurchase (within 30 days before or after the sale) of the same or substantially similar security. We permit wash sales as highly correlated substitute securities, that are not considered substantially similar, typically exist in most stock markets allowing an investor to re-establish a position with a similar risk-return profile after a capital loss. For an analysis of possible portfolio effects of wash sales when adequate substitute securities do not exist, see Jensen and Marekwica (2011). A shorting against the box transaction involves short selling securities that the investor owns to defer tax on capital gains. The Taxpayer Relief Act of 1997 no longer allows to delay taxation through shorting.

[8] To isolate the effect of the LUL assumption, we abstract away from investing in mutual funds where unrealized capital gain concerns can also be important. Like mutual funds, ETFs must pass unrealized capital gains onto investors generated by portfolio rebalancing. However, many ETFs substantially reduce and in some cases eliminate unrealized capital gains. This is achieved through a "redemption in kind" process described in Poterba and Shoven (2002).

endowment. The utility function for consumption and terminal wealth is of the constant relative risk aversion form with a relative risk aversion coefficient γ. Using the principle of dynamic programming, the Bellman equation for the investor's optimization problem, derived in Appendix A, can be solved numerically by backward induction starting at time T. We use a test region iterative contraction method introduced in Yang and Tompaidis (2013), to solve our consumption and investment problem with multiple stocks and several endogenous state variables due to capital gain taxation. The numerical algorithm is described in Appendix B.

3 A Two Date Example

In this section we return to the two trading date example described in the introduction to highlight the role the limited use of capital losses plays in determining an investor's optimal trading strategy.

Consider that the investor lives with probability one until $T = 2$ and maximizes expected utility of final period wealth over CRRA preferences with a coefficient of relative risk aversion equal to 5. The investor trades in one non-dividend paying stock and a riskless money market. Over time, he pays taxes on the money market's interest payment as well as capital gain taxes on the stock. At T, the portfolio is liquidated and all after-tax wealth is consumed. In this example, no capital gain tax liabilities are forgiven at time T. Endowment consists of one share of stock with a pre-existing tax basis-to-price ratio, $b(0)$, that is varied to capture different tax trading costs. When the $t = 0$ tax basis-to-price ratio is set lower (higher) than one, the investor has a capital gain (loss) in his position.

Using the same notation as Appendix A, the price system parameters are $S_0(0) = S_1(0) = 1$, $r = 0.05$, $\mu = 0.08$, and $\sigma = 0.16$, where S_0 and S_1 denote the money market and stock price, respectively. The stock price evolves as a binomial tree, so the investor will make a portfolio choice decision at $t = 0$ and $t = 1$ conditional on the stock price going up or down. The rate of appreciation (depreciation) of the stock over one time period is set at $e^{\sigma} = e^{0.16} = 1.174$ ($e^{-\sigma} = e^{-0.16} = 0.852$). The continuously-compounded expected stock return $\mu = 0.08$ determines the probabilities in the binomial tree. Tax rates are $\tau_I = 0.35$ and $\tau_C = 0.3$. The range for the investor's endowed basis-to-price ratio $b(0)$ is $[0.73, 1.38]$. This range corresponds to the lowest and highest stock price at T and captures the relative importance of the capitalization and lock-in effects.

Figure 2 summarizes the evolution of the optimal portfolio choice expressed as an equity-to-wealth ratio $\overline{\pi}$ (top three plots in the left panel) and the capital gain taxes paid Φ_{CG} (top three plots of the middle panel and all plots in the right panel). All plots are functions of the initial basis-to-price ratio $b(0)$. Portfolio choice decisions are made at times $t = 0$ and $t = 1$, while capital gain taxes are potentially paid at times $t = 0$, $t = 1$, and $t = 2$. In each plot, the solid line corresponds to the LUL case, the dashed line corresponds to the FUL case, and the dotted line corresponds to the NCGT case.

From the dotted lines in the equity-to-wealth plots of Figure 2, a NCGT investor always holds an equity-to-wealth ratio of approximately 0.43. To maintain this constant fraction, the investor trades the stock each period. At $t = 0$, the investor reduces his position from 1 share to 0.43 shares given the stock price is initially one; the proceeds of selling 0.57 shares are invested in the money market. At $t = 1$, when the stock price increases, the investor's fraction of wealth in equity rises above its optimum. The investor then reduces his equity-to-wealth ratio back to 0.43 by selling shares of stock and investing the proceeds in the money market. When the stock price decreases at $t = 1$, the investor is underexposed to equity and buys shares by selling part of the money market investment to again reach an equity-to-wealth ratio of 0.43.

With capital gain taxes, the investor can no longer costlessly trade leading to significant deviations from the NCGT case. However, the LUL trading strategy is considerably more sensitive to tax trading costs relative to the FUL trading strategy as can be seen in the first three plots in the left panel of Figure 2. This greater sensitivity is driven by the lack of tax rebates in the LUL case which impacts the optimal trading strategy across a broad range of basis-to-price ratios.

For a large enough basis-to-price ratio ($b(0) \geq 1.15$), the capitalization and lock-in effects are irrelevant as the LUL investor optimally trades as if he is the NCGT investor. In this region, realized capital losses at time $t = 0$ are large enough to cover any possible future capital gain taxes as shown in the Figure 2 tax plots. The optimal FUL trading strategy in this region is considerably different as the FUL investor holds more equity than the NCGT case. This extra equity demand is driven by the artificial tax rebate collected at $t = 0$ and possibly in the future if the stock price falls as shown in the tax plots. For the FUL investor, tax rebates act to truncate the down-side risk of holding equity, which elevates the demand.

For a basis-to-price ratio below 1.15, the LUL investor faces capital gain taxes when trading. When the basis-to-price ratio $b(0)$ is between 1.07 and 1.15, the LUL investor still never pays any capital gain taxes over his lifetime, but only by significantly reducing his equity-to-wealth ratio relative to the NCGT case. This captures a strong impact of the capitalization effect. When $b(0) = 1.07$, the

LUL investor's optimal equity-to-wealth ratio falls to 0.27 from 0.43. As the basis-to-price ratio falls toward 1.0, the LUL investor optimally holds more equity at $t = 0$, but still far below the NCGT benchmark. For the FUL investor, the ability to collect tax rebates through tax loss selling highly skews his portfolio choice as his optimal equity-to-wealth ratio is above the NCGT case. Additionally, the tax rebate artificially inflates his $t = 0$ wealth $W(0)$ as seen in the bottom left plot of Figure 2. Given the FUL investor's equity-to-wealth ratio is above the NCGT case and his wealth is elevated, his dollar investment in equity is also significantly higher than the NCGT case.

Tax trading costs at $t = 0$ matter for the LUL investor when the basis-to-price ratio falls below 1.0. The lock-in effect now becomes more important in addition to the capitalization effect. Given the initial endowment is one share of stock, the LUL investor is grossly over-exposed to equity from a risk-return perspective. When the basis-to-price ratio $b(0)$ is close to one, the LUL investor trades to an equity position significantly below the NCGT benchmark. Given he no longer has capital losses to shield future taxes, the after-tax benefit of holding stocks is still greatly reduced. As the basis-to-price ratio falls, the tax cost of trading at time $t = 0$ begins to dominate the benefit of holding less stock due to a risk-return motive leading the LUL investor to sell less equity. For the FUL investor, the probability of collecting tax rebates in the future still significantly skews his equity allocation since he continues to hold more than the NCGT benchmark. At the lowest initial basis-to-price ratio $b(0) = 0.73$, the FUL investor can never collect a tax rebate in the future. At this point, tax rebates no longer skew the FUL investor's trading strategy implying the LUL and FUL strategies converge.

This simple example highlights that the LUL investor's optimal trading strategy is sensitive to tax trading costs as captured by the basis-to-price ratio. If current capital losses are large enough to offset all future capital gain taxes, the LUL investor trades as if he is the NCGT investor. For small capital gains or losses embedded in the current portfolio, future taxes cannot be offset leading to a lower demand for equity than the NCGT investor through the capitalization effect. The FUL trading strategy does not reflect this sensitivity since equity demand is artificially elevated due to tax rebates.

4 Optimal Portfolios

We now consider a long-dated consumption-portfolio problem to understand quantitatively how the LUL trading strategy behaves. To highlight the conditional nature of the trading strategy, we characterize the structure of optimal portfolios at a particular time and state.

4.1 Parameterizations

In all parameterizations, the investor begins trading at age 20 and can live to a maximum of 100 years.[9] The investor has a time discount parameter $\beta = 0.96$. The bequest motive is set such that the investor plans to provide a perpetual real income stream to his heirs.

The return dynamics of the aggregate stock market are as follows: the expected return due to capital gains is $\mu = 8\%$, the dividend yield is $\delta = 2\%$, and the volatility is $\sigma = 16\%$. These dynamics are used when we study a single stock portfolio choice problem. For all parameterizations, the money market's return is $r_f = 5\%$. The rate of inflation, i, is set at 3% per annum.

When we study a two stock portfolio choice problem, both stocks are assumed to have identical expected returns, dividend yields, and volatilities. We allow the return correlation to vary and compute results for correlations $\rho = 0.4$, 0.8, and 0.9. To keep the pre-tax Sharpe ratio of an equally-weighted portfolio of these two stocks fixed across return correlations and equal to the aggregate stock market, each stock's dynamics are $\mu_n = 8\%$, $\delta_n = 2\%$, and $\sigma_n = \frac{\sigma}{\sqrt{0.5(1+\rho)}}$, where $n = 1, 2$.

Our base case choice of parameters, referred to throughout as the Base Case, studies portfolio problems with one and two stocks using the security return parameters just described. For the two stock case, we assume $\rho = 0.8$. The tax rates used are set to roughly match those faced by a wealthy investor under the U.S. tax code. We assume that interest is taxed at the investor's marginal income rate $\tau_I = 35\%$. Dividends are taxed at $\tau_D = 15\%$. The capital gain tax rate is set to the long-term rate $\tau_C = 20\%$.[10] To be consistent with the U.S. tax code, capital gain taxes are forgiven at the investor's death. The investor's relative risk aversion coefficient is set at $\gamma = 5$.

A way to increase the value of the FUL tax-loss selling option is to raise the capital gain tax rate allowing us to understand the impact on the capitalization effect. In the Capital Gain Tax 30% Case, the rate, for the one and two stock cases, roughly equals the 28% rate imposed after the U.S. 1986 Tax Reform Act. This rate also provides a setting that is roughly consistent with the long-term capital

[9]The probability that an investor lives up to period $t < T$ is given by a survival function, calibrated to the 1990 U.S. Life Table, compiled by the National Center for Health Statistics where we assume period $t = 0$ corresponds to age 20 and period $T = 80$ corresponds to age 100.

[10]The U.S. Tax Relief and Reform Act of 2003 changed several features of the tax code with respect to investments. Specifically, the long-term capital gain tax rate dropped from $\tau_C = 20\%$ to $\tau_C = 15\%$ for most individuals. Dividend taxation switched from being linked to investor's marginal income tax rate to a flat rate of $\tau_D = 15\%$. The 2006 Tax Reconciliation Act extended these rates to be effective until 2010. This was further extended through 2012 in legislation passed by Congress. From 2013, rates reverted to $\tau_C = 20\%$. Beginning in 2013 there is a contribution to Medicare of 3.8% for the lesser of the net investment income or the excess of the adjusted gross income above $200,000 ($250,000 for a joint return or half thereof if married taxpayers file separate returns). For a comprehensive summary of U.S. capital gain tax rates through time, see Figure 1 in Sialm (2009).

gain tax rate paid in many European countries. For example, the capital gain tax rates in Denmark, Finland, Norway and Sweden are currently 27%, 30%, 28%, and 30%, respectively. In 2009, Germany's individual capital gain tax rate rose to approximately 28% from 0%.[11]

In the two stock case, the Correlation 0.40 Case and the Correlation 0.90 Case model different diversification costs of not holding an equally-weighted stock portfolio. To illustrate a case where stock holdings decrease for the NCGT investor and hence the dollar value of tax-loss selling decreases for the FUL investor, the Higher Risk Aversion Case assumes that γ increases to 10. Finally, the No Tax Forgiveness at Death Case assumes capital gain taxes are assessed when the investor dies, a feature consistent with Canadian and European tax codes.

4.2 Benchmarks

To disentangle the role of the LUL assumption on portfolio choice, we focus on two benchmark portfolio choice problems. One benchmark is the case when the investor faces no capital gain taxation, abbreviated as NCGT. In this benchmark, the investor still pays dividend and interest taxes. Given the investment opportunity set is constant and the investor has CRRA preferences, the optimal trading strategy is to hold a constant fraction of wealth in each stock at all times. With the parameters in Subsection 4.1, the optimal portfolio choice is an overall equity-to-wealth ratio of 0.50 at all times. In the two stock case, this implies an equity-to-wealth ratio of 0.25 in each stock. Equity-to-wealth ratios are computed by dividing the value of the equity position by the value of the investor's wealth after subtracting consumption, and dividend and interest taxes, but before capital gains taxes.

Second, we also use the FUL case as a benchmark. Table 1 presents FUL optimal equity-to-wealth ratios ($\overline{\pi}(t)$) conditional on the beginning period equity-to-wealth and basis-to-price ratios ($\underline{\pi}(t)$ and $b(t)$), at ages 20 and 80. The FUL investor's trading strategy is remarkable when the basis-to-price ratio is greater than or equal to one as the tax rebate significantly skews the impact of the capitalization effect. For example, equity-to-wealth ratios at a basis-to-price ratio of one range from 14% (age 20) to 16% (age 80) higher than the NCGT benchmark. This additional demand for equity is driven by the collection of tax rebates that partially insure against drops in equity prices.

For basis-to-price ratios above one, equity-to-wealth ratios grow to 0.58 at age 20 and 0.60 at age 80, as more tax rebates induce higher income effects leading to higher investment in equity. Further,

[11] The German capital gain tax rate is 25% plus a church tax and tax to finance the five eastern states of Germany.

from the Capital Gain Tax 30% Case in the Internet Appendix we see that FUL equity-to-wealth ratios increase with the capital gain tax rate. This comparative static stresses that the FUL equity demand is indeed driven by tax rebates.

For basis-to-price ratios below one, the entering equity-to-wealth ratio becomes important through the lock-in effect in determining the optimal equity-to-wealth ratio for FUL investors. Specifically, the lock-in effect dominates other drivers of portfolio choice when the entering equity-to-wealth ratio is above the NCGT policy. For example, at a basis-to-price ratio of 0.5 equity-to-wealth ratios vary from 0.48 to 0.63 (0.50 to 0.70) almost linearly with entering equity-to-wealth ratios at age 20 (80).

4.3 Optimal Portfolios with the Limited Use of Capital Losses

4.3.1 One Stock

Table 1 presents LUL optimal equity-to-wealth ratios ($\overline{\pi}(t)$) conditional on the beginning period equity-to-wealth and basis-to-price ratios ($\underline{\pi}(t)$ and $b(t)$), at ages 20 and 80. All LUL optimal portfolios in this table have a zero carry-over loss entering the trading period. We see from the table that the LUL investor's trading strategy converges to the NCGT benchmark, independently of the entering equity-to-wealth ratio, when the basis-to-price ratio is significantly greater than one, as unused current capital losses virtually imply no capital gains taxes over the lifetime of the investor. When the basis-to-price ratio decreases towards one, equity-to-wealth ratios decline significantly below the NCGT benchmark independently of the entering equity-to-wealth ratio and age.

For basis-to-price ratios below one, we see three patterns. For beginning period equity-to-wealth ratios that are close to the NCGT benchmark, the LUL optimal equity-to-wealth ratio is close to that benchmark. Beginning period equity-to-wealth ratios that are below the NCGT benchmark lead to optimal equity-to-wealth ratio for the LUL investor that are significantly below the NCGT and the FUL benchmarks. With beginning period equity-to-wealth ratios above the NCGT benchmark, we find optimal equity-to-wealth ratios for the LUL investor that are only slightly below the FUL benchmark. However, the closer the basis-to-price ratio is to one, the larger is the difference between the equity allocation of the LUL investor and the FUL benchmark.

Table 1 also reports the percentage change in equity-to-wealth ratios of FUL versus LUL investor. The difference is always positive highlighting that FUL equity demand is elevated by tax rebates. The differences are economically large ranging from 0% to 24.1% at age 20 and from 0% to 23.1% at age

80.

4.3.2 Two Stocks

In the one stock setting in Table 1, equity positions can never simultaneously exhibit a capital gain and a capital loss. It is natural to ask how optimal investment strategies behave when capital gains and losses occur simultaneously. If enough realized capital gains are generated with multiple stocks, tax rebates might have a smaller impact on conditional trading strategies. The top panel in Table 2 shows the Base Case with two stocks for an age 80 investor who is overexposed to equity with initial allocations $\underline{\pi}_1 = 0.3$ and $\underline{\pi}_2 = 0.4$. This choice of an initial stock position allows us to quantify the tradeoff between minimizing tax-induced trading costs and holding the optimal mix of equity and the money market. As in the one stock case, all LUL portfolios in this table have a zero carry-over loss. Cases with a positive carry-over loss are well captured by examining trading strategies with basis-to-price ratios bigger than one entering the period.

When both stock positions have basis-to-price ratios greater than or equal to one, $b_1 \geqslant 1$ and $b_2 \geqslant 1$, the capitalization effect dominates and therefore the optimal trading strategies in each stock and at the portfolio level are similar to the one stock case. The LUL investor chooses to hold equal positions in each stock with a total equity position never greater than the NCGT benchmark, 0.25 in each stock, as seen in Table 2. Higher basis-to-price ratios, b_1 or b_2, increase simultaneously both allocations up to the NCGT benchmark.

When both stock positions have low basis-to-price ratios and the investor is overinvested in equity, the lock-in effect dominates, and the LUL optimal trading strategies are overall similar to the one stock case. Further, given tax forgiveness at death, the investor chooses to remain overexposed to equity at the portfolio level. Note, however, that the relation between the two basis-to-price ratios and their interplay with initial allocations, $\underline{\pi}_1 = 0.3$ and $\underline{\pi}_2 = 0.4$, matter for the portfolio composition. When we set $b_1 = b_2 < 1$, so that the stocks face identical lock-in effects, we see that the allocation to stock 1 is almost insensitive to its own basis-to-price ratio as the investor is much more overexposed to stock 2. Here the investor's desire to diversify drives reallocations. When we set $b_1 < b_2 < 1$ or $b_2 < b_1 < 1$ then only overexposure to the stock with the smaller lock-in effect is reduced. For example, when $b_1 = 0.6$ and $b_2 = 0.8$, the optimal trading strategy is to reduce the overexposure to stock 2 from 0.4 to 0.31 while keeping the overexposure to stock 1 at 0.3. With $b_1 = 0.8$ and $b_2 = 0.6$, the optimal trading strategy is to reduce the overexposure to stock 1 while keeping the overexposure to stock 2. For the case $b_2 < b_1 < 1$, the after trade difference in allocations between the two stocks increases relative to initial positions, which implies that the lock-in effect dominates the investor's desire to diversify.

The main benefit of examining the two stock case is that we can study how the optimal strategies behave when the investor simultaneously has an embedded capital gain and loss in the portfolio, $b_1 > 1$ and $b_2 < 1$ or $b_1 < 1$ and $b_2 > 1$. Consider for example the case when the investor's entering equity positions have basis-to-price ratios of $b_1 = 1.2$ and $b_2 = 0.5$. Here the investor is overinvested in stock with a capital loss in stock 1 and a capital gain in stock 2. The LUL investor tax-loss sells his position in stock 1 and reestablishes a position of $\overline{\pi}_1 = 0.19$. Using his realized capital losses to offset the capital gain on stock 2, he reduces the stock 2 position to $\overline{\pi}_2 = 0.29$. Note that the cheaper it is to reduce the overinvestment in the stock with embedded capital gains, i.e., the closer the stock's relative basis is to one ($b_1 < 1$ or $b_2 < 1$), the smaller is the allocation to that stock, and the larger is the allocation to the stock with embedded losses to also optimize the total equity allocation.

Cases for which one stock has embedded gains and the other stock has a basis-to-price ratio of 1 are similar in that the stock that can be traded without incurring capital gain taxes is adjusted to optimize the total equity allocation. However, the stock with embedded gains remains at the initial allocation, $\underline{\pi}_1 = 0.3$ or $\underline{\pi}_2 = 0.4$, except maybe when its basis-to-price ratio is close to 1.

While we have modeled the multiple stock setting with only two stocks, these results should generalize to portfolios with many stocks. For any stock with an embedded loss, it is always optimal to liquidate the entire position to generate a realized capital loss. For stocks with embedded gains that an investor is overexposed to, any rebalancing will be small to minimize the capital gain taxes to be paid. Combining these two types of trades, several states of the world will occur where the investor's realized losses are bigger than the realized gains.

4.4 Robustness

4.4.1 One Stock

Table 3 explores three comparative static cases for the LUL investor: higher capital gain taxes, imposing capital gain taxes at death, and increasing the investor's risk aversion. From the Capital Gain Tax 30% Case, we see that the LUL optimal portfolios are below the Base Case at a basis-to-price ratio of one. For basis-to-price ratios above one, equity-to-wealth ratios for the Capital Gain Tax 30% Case are slightly below or equal to the Base Case. Equity-to-wealth ratios, with basis-to-price ratio below one, depend on entering equity-to-wealth ratios in a way that is comparable to the Base Case. Again, optimal equity allocations in the Capital Gain Tax 30% Case are smaller than the Base Case.

The No Tax Forgiveness at Death Case reports the optimal equity allocations when capital gain taxation is not forgiven at death. Specifically, with no tax forgiveness, optimal equity allocations no longer increase with age as in the Base Case. This can be seen by comparing equity-to-wealth ratios at age 20 with the ratios at age 80 in Table 3.

In the Higher Risk Aversion Case, the investor's risk aversion is increased to $\gamma = 10$ to capture a scenario where equity is a less important component of the investor's portfolio. The NCGT equity-to-wealth allocation is now 0.25, compared to 0.5 in the Base Case. We note that increasing risk aversion leads to largely the same features for optimal LUL portfolios as in the Base Case except at lower allocations.

4.4.2 Two Stocks

With the two stock case, we can examine the tradeoff between the tax cost of trading and the benefit of holding a well-diversified equity portfolio. First, we compare the Base Case to the Capital Gain Tax 30% Case in Table 2. Recall that the entering equity allocations ($\underline{\pi}_1 = 0.3$ and $\underline{\pi}_2 = 0.4$) overexpose the investor to equity and that the optimal allocation in an untaxed environment is $\overline{\pi}_1 = 0.25$ and $\overline{\pi}_2 = 0.25$. Thus, both positions should be reduced. When $b_1 < b_2 < 1$, both assets have embedded capital gains, but the second stock is cheaper to trade as the embedded capital gain is smaller. In the two top panels of Table 2, we see that the entering allocation of stock 2 is reduced, while the position in stock 1 remains unchanged in the entire area where $b_1 < b_2 < 1$. For the Capital Gain Tax 30% Case, the reduction in exposure of the entering portfolio to equity is smaller since the tax trading costs are higher. The case when $b_2 < b_1 < 1$ is similar to $b_1 < b_2 < 1$ except that now it is cheaper to trade in stock 1. When $b_1 = b_2 < 1$ the tax related trading costs are identical across the two stocks, however, the entering position in stock 2 is further away from the NCGT benchmark. Thus, the allocation to stock 2 is reduced first as the common basis increases from 0.5 to 0.9. The lock-in effect is stronger when the capital gain tax rate is higher, and the Capital Gain Tax 30% Case shows that it is not optimal to reduce the allocation in stock 1 even when $b_1 = b_2 = 0.9$. Thus, with embedded capital gains, the higher capital gain tax leads to a higher total equity allocation and higher allocation in stock 2 relative to the Base Case.

In the two bottom panels of Table 2, we change the return correlation between the two stocks to 0.40 and 0.90 to capture different diversification costs relative to the Base Case correlation of 0.80. When the correlation increases, diversification benefits are less important implying the investor is willing to hold a less diversified position when it is costly to trade. This is evident, for example, by comparing the optimal stock 2 position as the stock 2 basis-to-price ratio varies, but the stock 1 basis-to-price ratio is fixed at 1. From a tax perspective, stock 1 can be traded at no cost; however, stock 2 is costly to trade if its basis-to-price ratio is less than 1. In this situation, the investor facing a return correlation of 0.4 is more willing to reduce the stock 2 position from 0.4 than the investor facing a return correlation of 0.9. Importantly, the return correlation does not have a large impact on LUL strategies.

5 The Lifetime Structure of Optimal Wealths and Portfolios

To gain insights into the evolution of the optimal strategy, including the wealth distribution over an investor's lifetime, we perform Monte Carlo simulations using our numerical solution of the optimal consumption and portfolio policies. The investor starts with no embedded capital gains at age 20 and an initial wealth of \$100. We track the evolution of the investor's optimal portfolio over time conditional on the investor's survival. All simulations are over $50,000$ paths.

The results of our Monte Carlo simulations are reported in Table 4. These are one stock simulations for the LUL case with and without forced liquidation. We use 0.1 and 0.2 for liquidation shock arrival rates (Constantinides (1983)) and consider full and partial liquidation. In the table, the top panel presents the Base Case, while the bottom panel presents the Capital Gain Tax 30% Case. Each panel reports characteristics of the LUL portfolio choice problem at ages 40, 60, and 80. For each quantity, a selection of the percentiles of the distribution, the mean, and the standard deviation are reported. The column labeled Wealth gives the investor's current financial wealth expressed in dollars. The columns labeled Equity-to-Wealth Ratio and Basis-to-Price Ratio present the characteristics of the optimal equity position. The Cumulative Capital Gain Tax-to-Wealth Ratio column presents the undiscounted cumulative taxes paid from age 20 to the current age divided by the wealth at the current age. Finally, the column LUL Carry-Over Loss-to-Wealth Ratio presents the carry-over loss variable at the current age.[12]

The simulations demonstrate that the optimal wealths are significantly impacted by liquidations. From Table 4, we see that the investor's wealth without liquidation shocks (No LS) is higher at each percentile and age across all cases relative to the investor's wealth with liquidations (LS). For example in the Base Case at age 80, the mean wealth without liquidation shocks is more than twice as large as with liquidation shocks. This mean wealth difference grows to threefold in the Capital Gain Tax 30% Case. The difference is driven by the tax costs of liquidation shocks and a reduced allocation to equity due to the resetting of locked-in stock positions. In the Base Case at age 80, the mean equity-to-wealth ratio without liquidation shocks is 0.7 while the mean allocation with liquidation is only 0.52. In the Capital Gain Tax 30% Case, the mean of the No LS allocation grows to 0.72 but reaches only 0.52 when liquidation occurs. The higher wealth in the Capital Gain Tax 30% Case without liquidation is driven by a stronger lock-in effect which significantly reduces the mean cumulative capital gain tax-to-wealth ratio relative to the Base Case. More realistically, in the LS case the mean cumulative capital gain tax-to-wealth ratio increases from the Base Case to the Capital Gain Tax 30% Case.

Importantly, from the simulations we learn that the LUL investor faces large embedded gains as can be seen from the small mean basis-to-price ratios. The investor's carry-over loss-to-wealth ratio column also demonstrates this feature as the carry-over loss variable is only nonzero early in the investor's life. Similar behavior occurs in several of the other one stock and two stock simulations (see the Internet Appendix) when there is no liquidation over the life of the investor. However, with liquidation shocks, the carry-over loss variable is nonzero not just early in the investor's life but remains so until old age.

Empirical evidence summarized in Poterba (2002) shows that investors do realize capital gains and hence pay capital gain taxes. The life-cycle structure of the cumulative capital gain tax-to-wealth ratios in the Base Case and the Capital Gain Tax 30% Case with liquidation shocks seem more consistent with the evidence in Poterba (2002) than the cases without liquidation shocks. In this sense, we expect the carry-over loss variable of average as well as wealthy investors to remain nonzero for a longer time than suggested by the evolution of optimal strategies in the No LS cases.

[12]Each mean estimate's standard error can be computed by dividing the Monte Carlo standard deviation given in the table by $\sqrt{50,000} = 223.6$.

6 The Economic Costs of the LUL and the FUL Cases

6.1 Wealth Equivalents

To quantify the economic significance of capital gain taxation, we compare the FUL and the LUL investors to an untaxed investor. Specifically, Table 5 reports the wealth equivalent change that keeps an age 20 NCGT investor indifferent to being capital gain taxed under the LUL and FUL cases, respectively. A positive (negative) percentage wealth equivalent change denotes that the NCGT investor's welfare improves (worsens) by paying a capital gain tax.[13] We present results for the one stock Base Case and the Capital Gain Tax 30% Case. Results for the two stock cases are summarized in the Internet Appendix.

We consider five cases: no liquidation shock, a liquidation shock with a 0.2 arrival rate and a liquidation fraction 0.5, a liquidation shock with 0.2 arrival rate and a liquidation fraction 0.75, a liquidation shock with 0.2 arrival rate and a liquidation fraction 1, and a liquidation shock arrival every period and a liquidation fraction 1. The fifth case assumes that each period all capital gain taxes and losses are realized.

The wealth equivalent change analysis further illustrates that tax rebates are an important driver of an FUL investor's optimal portfolio choice. For all one and two stock cases without liquidation except the No Tax Forgiveness at Death Case, the FUL wealth equivalent changes are positive (see Internet Appendix). Specifically, in Table 5 we see that the FUL no liquidation shock and liquidation shock with 0.2 arrival and liquidation fraction 0.5 cases have wealth equivalent changes that are positive. In contrast, the LUL wealth equivalent changes are always negative. Hence, a NCGT investor is actually never better off paying capital gain taxes under the LUL scenario, which seems natural. Importantly, the economic costs between LUL and FUL increase from slightly above two and a half percent for the no liquidation Base Case ($2.16\% - (-0.45\%)$) to six percent for the Capital Gain Tax 30% Case ($-7.50\% - (-13.49\%)$) with liquidation shock arrival every period and liquidation fraction 1. These examples highlight that the economic cost of restricting the use of capital losses is large.

[13]Note that our measure of the cost of taxation is in contrast to most of the existing literature (Constantinides (1983); Dammon, Spatt, and Zhang (2001b); Garlappi, Naik, and Slive (2001)) as we do not measure tax costs relative to an accrual-based capital gain taxation system where all gains and losses are marked-to-market annually. Instead, our wealth equivalent change measure is meant to capture the change in an investor's welfare by imposing a capital gain taxation scheme. In particular, this measure allows us to capture how undervalued the capitalization effect is under an FUL-based tax system.

6.2 Backtesting

An alternative way to quantify the economic costs or differences between the FUL and LUL cases is to backtest the portfolio choice problems. For this exercise, we collect annual time series of the S&P 500 Index, its dividends, the one year rate of interest as proxy for the riskless rate of interest, and inflation from Robert Shiller's website. From the series, we estimate historical means and the standard deviations of equity returns for two periods: $1927 - 2006$ and $1950 - 2011$.[14] To simplify, all tax rates are set as in the Base Case instead of historical averages.[15] The backtests start at 1927 or 1950, when the investor is 20 years old with initial stock position of 0, initial tax basis of 1, and initial carry-over loss of 0. For the No LS shock case, we assume no liquidation shock over the investor's lifetime from age 20 to age 99 while in the LS case, we assume that the investor experiences a one-time 100% liquidation shock at age 50. Using these parameters and the S&P 500 Index return data we find the investors' optimal FUL and LUL strategy over their lifetime.

Figure 3 shows the evolution of the FUL and LUL equity allocations for the $1927 - 2006$ and $1950 - 2011$ time periods with and without liquidation at age 50. The top left plot shows the after-tax FUL (solid line) and LUL (dashed line) No LS strategies, $\overline{\pi}$, expressed as an equity-to-wealth ratio. We see that the LUL strategy is 0.12 below the FUL strategy in 1927. The difference between the strategies reduces to roughly 0.05 in the early 1960s and drops below 0.02 after 1980. From the LS case in the top right plot, we learn that after the liquidation shock the difference between the equity-to-wealth ratios increases to above 0.1. While by the early 1980s, the difference is of the same magnitude as without a liquidation shock, we note that the LUL strategy is below the FUL strategy by more than 0.05 almost until 1980. When the investment strategy starts in the 1950, the middle plots of Figure 3, we see comparable patterns. Without a liquidation shock, the strategies converge roughly after 40 years, while with a liquidation shock, the LUL and FUL strategies may never converge over the lifetime of the investors.

To mitigate cherry-picking concerns, in the bottom plots of Figure 3, we average over the equity allocations of 55 investors from age 20 to age 50. In this rolling window type of backtest the first investor experiences the return series from 1927 to 1957, the second investor starts in 1928, while

[14] The average log equity return equals 5.83% (7.01%) with a 18.86% (15.63%) standard deviation, a 4.17% (3.56%) dividend yield, the annual rate of interest on average is 4.64% (5.52%), and on average inflation (based on CPI) reached 3.06% (3.66%) over the period $1927 - 2006$ ($1950 - 2011$).

[15] Note that it is difficult to anticipate changes in the tax code plus incorporating time variation in tax rates goes beyond the goals of this article.

the last investor enters in 1981 and reaches age 50 in 2011. These plots support our view that the backtests shown in the top and middle plots of Figure 3 are conservative and that the difference between the FUL and LUL allocations are reliably economically significant independently of whether investors experience liquidation shocks or not.

In the Internet Appendix accompanying this paper, we consider additional backtests where, in addition to using the realized S&P 500 Index returns, we use realized dividends, inflation rates, and interest rates. We also consider the Capital Gain Tax 30% Case and tests starting in 1980. In all cases, the differences between the FUL and LUL strategies are large and may never converge over the lifetime of the investors.

7 Conclusion

We integrate the *limited use of losses* assumption into multiple stock portfolio problems with capital gain taxation. Requiring that capital losses can only be used to offset current or future realized gains significantly changes the after-tax risk-return tradeoff of holding equity. With small embedded gains or losses in an existing portfolio when the capitalization effect is most important, an investor holds significantly less equity. If embedded capital losses are large enough, it is optimal to trade the untaxed investor's strategy. When embedded capital gains are large, the capital gain lock-in effect dominates making it difficult for an investor to trade out of a large equity position.

In contrast, a *full use of losses* investor's trades are artificially impacted by tax rebates. These tax rebates act as an income process that pays off in down markets leading to a misleading higher demand for equity relative to an untaxed investor when capital gains are not too large in the existing portfolio. Tax rebates greatly skew optimal wealths, collected taxes, and total dollar investment in equity over an investor's life. The motives for capturing tax rebates are strong enough to generate a counterfactual welfare result — the full use of losses investor prefers to pay capital gain taxes rather than being untaxed.

In one extension of our framework, we use periodic liquidation shocks to lessen the unrealistically strong capital gain lock-in effect over the lifetime of an investor. Other modifications to the limited use of losses portfolio problem that would further lessen the capital gain lock-in effect by making low basis-to-price ratios less likely include modeling a price system with mean-reverting dynamics and incorporating an income process with borrowing constraints that would lead to equity investment occurring through time. We leave these extensions to future research.

A Investor Consumption-Portfolio Problem Description

The mathematical description of the portfolio problem outlined in Section 2 is now presented. Our multiple risky stock model is based on the single stock setting of Dammon, Spatt, and Zhang (2001b) and the multiple stock setting of Gallmeyer, Kaniel, and Tompaidis (2006) where our notation and setup mainly follows from the latter. The major difference here relative to Gallmeyer, Kaniel, and Tompaidis (2006) is that our work incorporates the limited use of capital losses with no short selling.

A.1 Security Market

The economy is discrete-time with trading dates $t = 0, ..., T$. The investor trades each period in a riskless money market and N risky stocks. For simplicity, we consider a constant opportunity set.

The riskless money market has a time t price of $S_0(t)$ and pays a continuously compounded pre-tax interest rate r. The money market's price dynamics are given by

$$S_0(t + \Delta_t) = S_0(t) \exp\left(r\Delta_t\right), \tag{A.1}$$

where Δ_t is an arbitrary time interval.

Stock market investment opportunities are represented by N stocks each with a time t ex-dividend price $S_n(t)$ for $n = 1, \ldots, N$. Each stock pays a pre-tax dividend of $\delta_n S_n(t)$ at time t where δ_n is stock n's dividend yield. Stock n's pre-tax ex-dividend price follows a lognormal distribution with price dynamics over the time interval Δ_t given by

$$S_n(t + \Delta_t) = S_n(t) \exp\left(\left(\mu_n - \frac{1}{2}\sigma_n^2\right)\Delta_t + \sigma_n\sqrt{\Delta_t}\tilde{z}_n\right), \tag{A.2}$$

where $\tilde{z}_n$ is a standard normal distribution. The quantity μ_n is the instantaneous capital gain expected growth rate and σ_n is the instantaneous volatility of the stock. The shocks $\tilde{z}_n$ for $n = 1, \ldots, N$ have a variance-covariance matrix Σ inducing a correlation structure across stocks. To match the yearly trading interval of the investor in our economy, we assume that $\Delta_t = 1$ year.

A.2 Investor's Problem

Given a discrete-time economy with trading dates $t = 0, \ldots, T$, an investor endowed with initial wealth in the assets chooses an optimal consumption and investment policy in the presence of realized capital gain taxation. The investor lives for at most T periods and faces a positive probability of death each period. The probability that an investor lives up to period t, $0 < t < T$, is given by the survival function $H(t) = \exp\left(-\sum_{s=0}^{t}\lambda_s\right)$ where λ_s is the single-period hazard rate for period s where we assume $\lambda_s > 0$, $\forall s$, and $\lambda_T = \infty$. This implies $0 \leq H(t) < 1$, $\forall\, 0 \leq t < T$. At T, the investor exits the economy, implying $H(T) = 0$. We assume that the investor makes annual decisions starting at age 20 corresponding to $t = 0$ with certain exit from the economy at age 100 implying $T = 80$. The hazard rates λ_s are calibrated to the 1990 U.S. Life Tables compiled by the National Center for Health Statistics to compute the survival function $H(t)$ from ages 20 ($t = 0$) to 99 ($t = 79$).

The trading strategy from time t to $t+1$ in the money market and the stocks is given by $(\alpha(t), \theta(t))$ where $\alpha(t)$ denotes the shares of the money market held and $\theta(t) \equiv (\theta_1(t), \ldots, \theta_N(t))^\top$ denotes the vector of shares of stocks held where an individual element $\theta_n(t)$ denotes the holding of stock n.

A.2.1 Interest and Dividend Taxation

The investor faces three forms of taxation in our analysis: interest taxation, dividend taxation, and capital gain taxation. Interest income is taxed as ordinary income at the constant rate τ_I, while dividend income is taxed at the constant rate τ_D. The total taxes paid on interest and dividend income at time t are

$$\Phi_{I,D}(t) = \tau_I \, \alpha(t-1) \, S_0(t-1) \, (\exp(r) - 1) + \tau_D \sum_{n=1}^{N} \theta_n(t-1) S_n(t) \delta_n. \tag{A.3}$$

If the investor dies at time t, interest and dividend taxes are still paid.

A.2.2 Capital Gain Taxation

Using our two definitions of capital gain taxation, realized capital gains and losses in the stock are subject to a constant capital gain tax rate of τ_C. Computing the capital gain taxes due each period requires keeping track of the past purchase prices of each stock which forms that stock's tax basis. The tax basis for each stock is calculated as a weighted-average purchase price. Let $B_n(t)$ denote the nominal tax basis of stock n after trading at time t. The stock basis evolves as

$$B_n(t) = \begin{cases} S_n(t) & \text{if } \theta_n(t) = 0 \text{ or } \frac{B_n(t-1)}{S_n(t)} > 1, \\ \frac{B_n(t-1)\,\theta_n(t-1) + S_n(t)\,(\theta_n(t) - \theta_n(t-1))^+}{\theta_n(t-1) + (\theta_n(t) - \theta_n(t-1))^+} & \text{otherwise,} \end{cases} \tag{A.4}$$

where $x^+ \stackrel{\triangle}{=} \max(x, 0)$. If $\theta_n(t) = 0$, the basis resets to the current stock price, $B_n(t) = S_n(t)$. Here we have assumed that the investor is precluded from short-selling stock n.

Under the FUL case, any realized capital gains or losses are subject to capital gain taxation. The capital gain taxes $\Phi_{CG}^{FUL}(t)$ at time t under the FUL case are

$$\Phi_{CG}^{FUL}(t) = \tau_C \left(\sum_{n=1}^{N} (S_n(t) - B_n(t-1))^+ (\theta_n(t-1) - \theta_n(t))^+ - \sum_{n=1}^{N} (B_n(t-1) - S_n(t))^+ \theta_n(t-1) \right), \tag{A.5}$$

where the first term calculates taxes from selling stocks with capital gains and the second term calculates reductions in taxes through capital losses from tax-loss selling. If death occurs at some time t', all capital gain taxes are forgiven implying $\Phi_{CG}^{FUL}(t') = 0$.

While the FUL case allows for negative taxes or a tax rebate when capital losses are realized, the LUL case eliminates all tax rebates. Realized capital losses can only be used to offset current or future realized capital gains. As a result, an additional state variable, the accumulated capital loss $L(t)$, is required. This state variable measures accumulated unused realized capital losses as of time t and evolves as

$$L(t) = \left(L(t-1) + \sum_{n=1}^{N} (B_n(t-1) - S_n(t))^+ \theta_n(t-1) - \sum_{n=1}^{N} (S_n(t) - B_n(t-1))^+ (\theta_n(t-1) - \theta_n(t))^+ \right)^+. \tag{A.6}$$

The accumulated capital loss $L(t)$ is modeled as a nonnegative state variable. A positive value is interpreted as unused realized capital losses. The first summation in (A.6) captures any increase in accumulated capital losses due to tax-loss selling. Based on Gallmeyer and Srivastava (2011), the investor is always weakly better off realizing all capital losses today even if he cannot use them immediately. This feature simplifies our analysis in that extra state variables are not needed that track capital losses still inside the portfolio. The second summation in (A.6) captures any decline in accumulated capital losses that are used to offset realized capital gains when shares are sold at time t. The max operator is applied to the entire expression as it is possible that realized sales with capital gains may extinguish all unused capital losses.

Under the LUL case, only realized capital gains are subject to capital gain taxation. Realized capital losses are used to offset future realized gains. The capital gain taxes $\Phi_{CG}^{LUL}(t)$ at time t under the LUL case are

$$\Phi_{CG}^{LUL}(t) = \tau_C \Bigg(\sum_{n=1}^{N} (S_n(t) - B_n(t-1))^+ (\theta_n(t-1) - \theta_n(t))^+ \\ - \sum_{n=1}^{N} (B_n(t-1) - S_n(t))^+ \theta_n(t-1) - L(t-1) \Bigg)^+, \quad \text{(A.7)}$$

where capital gain taxes are paid when the investor realizes capital gains and does not have large enough accumulated capital losses $L(t-1)$ or current realized capital losses to offset that gain. If death occurs at some time t', all capital gain taxes are forgiven implying $\Phi_{CG}^{LUL}(t') = 0$.

A.2.3 Trading Strategies

We now define the set of admissible trading strategies when the investor can invest in the stock and the riskless money market. Again, we assume that the investor is prohibited from shorting any security.

The quantity $W(t+1)$ denotes the time $t+1$ wealth before portfolio rebalancing and any capital gain taxes are paid, but after dividend and interest taxes are paid. It is given by

$$W(t+1) = \alpha(t)\, S_0(t)\, ((1-\tau_I)\exp(r) + \tau_I) + \sum_{n=1}^{N} S_n(t+1)\, (1 + \delta_n(1-\tau_D))\theta_n(t), \quad \text{(A.8)}$$

where (A.3) has been substituted. Given that no resources are lost when rebalancing the portfolio at time t, $W(t)$ is given by

$$W(t) = \alpha(t) S_0(t) + \sum_{n=1}^{N} S_n(t)\theta_n(t) + C(t) + \Phi_{CG}^{j}(t), \quad j \in \{FUL, LUL\}, \quad \text{(A.9)}$$

where $C(t) > 0$ is the time t consumption.

Substituting (A.9) into (A.8) gives the dynamic after-tax wealth evolution of the investor,

$$W(t+1) = \left(W(t) - \sum_{i=1}^{N} S_n(t)\theta_n(t) - C(t) - \Phi_{CG}^{j}(t) \right) ((1-\tau_I)\exp(r) + \tau_I) \\ + \sum_{n=1}^{N} S_n(t+1)\, (1 + \delta_n(1-\tau_D))\theta_n(t), \quad j \in \{FUL, LUL\}. \quad \text{(A.10)}$$

Additionally, the investor faces a margin constraint modeled as in Gallmeyer, Kaniel, and Tompaidis (2006). The margin constraint imposes a lower bound on the dollar amount of borrowing in the money market,

$$\alpha(t)\, S_0(t) \geq -(1 - m_+) \sum_{n=1}^{N} S_n(t)\theta_n(t), \quad \text{(A.11)}$$

where $1 - m_+$ denotes the fraction of equity that is marginable. Throughout, we use $m_+ = 0.5$ which is consistent with Federal Reserve Regulation T for initial margins.

An *admissible trading strategy* is a consumption and a security trading policy (C, α, θ) such that for all t, $C(t) \geq 0$, $W(t) \geq 0$, $\theta(t) \geq 0$, and (A.10)-(A.11) are satisfied. The set of admissible trading strategies is denoted $\mathcal{A}$.

A.2.4 Investor's Objective

The investor's objective is to maximize his discounted expected utility of real lifetime consumption and final-period wealth at the time of death by choosing an admissible trading strategy given an initial endowment. If death occurs on date t, the investor's assets totaling $W(t)$ are liquidated and used to purchase a perpetuity that pays to his heirs a constant real after-tax cash flow of $R^* W(t)$ each period starting on date $t+1$. The quantity R^* is the one-period after-tax real riskless interest rate computed using simple compounding. In terms of the instantaneous nominal riskless money market rate r and the instantaneous inflation rate i, R^* is defined by

$$R^* = ((1-\tau_D)\exp(r) + \tau_D)\exp(-i) - 1.$$

Under the assumption that the investor and his heirs have identical preferences of the constant relative risk aversion (CRRA) form with a coefficient of relative risk aversion of γ and a common time preference parameter β, the investor's optimization problem is given by

$$\max_{(C,\alpha,\theta)\in\mathcal{A}} E\Bigg[\sum_{t=0}^{T} \beta^t \Bigg\{ \frac{H(t)}{1-\gamma}\,(\exp(-it)C(t))^{1-\gamma} + \frac{H(t-1)-H(t)}{1-\gamma} \sum_{s=t+1}^{\infty} \beta^{s-t}\,(\exp(-it)R^* W(t))^{1-\gamma} \Bigg\}\Bigg]. \quad \text{(A.12)}$$

The objective function captures the expected utility of future real consumption as well as the bequest motive to the investor's heirs.

Since $\sum_{s=t+1}^{\infty} \beta^{s-t} = \frac{\beta}{1-\beta}$, the investor's objective function simplifies, leading to the optimization problem

$$\max_{(C,\alpha,\theta)\in\mathcal{A}} E\Bigg[\sum_{t=0}^{T} \beta^t \Bigg\{ \frac{H(t)}{1-\gamma}\,(\exp(-it)C(t))^{1-\gamma} + \frac{H(t-1)-H(t)}{1-\gamma}\,\frac{\beta}{1-\beta}\,(\exp(-it)R^* W(t))^{1-\gamma} \Bigg\}\Bigg]. \quad \text{(A.13)}$$

A.3 Change of Variables

As in a no-tax portfolio choice problem with CRRA preferences, the optimization problem (A.13) is homogeneous in wealth, and thus independent of the investor's initial wealth. To show that wealth is not needed as a state variable when solving (A.13), we express the optimization problem's controls as being proportional to time t wealth $W(t)$ before security trading but after the payment of taxes on dividends and interest. We define

$$\underline{\pi}_n(t) \triangleq \frac{S_n(t)\theta_n(t-1)}{W(t)}, \qquad \overline{\pi}_n(t) \triangleq \frac{S_n(t)\theta_n(t)}{W(t)}, \quad \text{(A.14)}$$

where $\underline{\pi}_n(t)$ and $\overline{\pi}_n(t)$ are the proportions of stock n owned entering and leaving period t, with respect to time t wealth $W(t)$. Note that the investor will never choose a trading strategy that leads to a non-positive wealth at any time given our utility function choice, the bequest motive, and the positive probability of death over each period. Hence, portfolio weights are well-defined as $W(t) > 0$ for all t.

Using (A.14), it is useful to express each stock's basis $B_n(t)$ as a basis-price ratio $b_n(t+1) \triangleq \frac{B_n(t)}{S_n(t+1)}$.

Using (A.4), the basis-price ratio evolves as

$$b_n(t+1) = \begin{cases} \frac{S_n(t)}{S_n(t+1)} & \text{if } \underline{\pi}_n(t) = 0 \text{ or } b_n(t) > 1, \\ \frac{b_n(t)\underline{\pi}_n(t) + (\overline{\pi}_n(t) - \underline{\pi}_n(t))^+}{\frac{S_n(t+1)}{S_n(t)}(\underline{\pi}_n(t) + (\overline{\pi}_n(t) - \underline{\pi}_n(t))^+)} & \text{otherwise.} \end{cases} \tag{A.15}$$

If $\overline{\pi}_n(t) = 0$, the basis-price ratio $b_n(t+1)$ resets to the ratio of the time t and $t+1$ stock n price, $b_n(t+1) = \frac{S_n(t)}{S_n(t+1)}$. The basis-price ratio at time $t+1$ can be expressed as a function of the capital gain of stock n over one period $\frac{S_n(t+1)}{S_n(t)}$, the previous period's basis-price ratio $b_n(t)$, and the equity proportions $\overline{\pi}_n(t)$ and $\underline{\pi}_n(t)$.

For the LUL case, the accumulated loss state variable $L(t)$ must also be expressed proportional to $W(t)$. Similar to the stock position, we define

$$\underline{l}(t) \triangleq \frac{L(t-1)}{W(t)}, \qquad \bar{l}(t) \triangleq \frac{L(t)}{W(t)}, \tag{A.16}$$

where $\underline{l}(t)$ and $\bar{l}(t)$ are the proportions of accumulated capital losses entering and leaving period t, with respect to time t wealth $W(t)$.

Using (A.6), the proportional accumulated capital losses evolve as

$$\bar{l}(t) = \left(\underline{l}(t) + \sum_{n=1}^{N}(b_n(t)-1)^+\underline{\pi}_n(t) - \sum_{n=1}^{N}(1-b_n(t))^+(\underline{\pi}_n(t) - \overline{\pi}_n(t))^+\right)^+. \tag{A.17}$$

Note that this quantity is independent of wealth $W(t)$.

Using the equity proportions, the basis-price ratios, and the proportional accumulated capital losses, the total capital gain taxes paid at time t, $\Phi^i_{CG}(t)$, can be written proportional to $W(t)$. Expressing $\Phi^i_{CG}(t) = W(t)\,\phi^i_{CG}(t)$, where $i \in \{FUL, LUL\}$, we obtain that $\phi^i_{CG}(t)$ is independent of $W(t)$. For the FUL case,

$$\phi^{FUL}_{CG}(t) \;\; = \;\; \tau_C\left(\sum_{n=1}^{N}(1-b_n(t))^+(\underline{\pi}_n(t) - \overline{\pi}_n(t))^+ - \sum_{n=1}^{N}(b_n(t)-1)^+\underline{\pi}_n(t)\right). \tag{A.18}$$

For the LUL case,

$$\phi^{LUL}_{CG}(t) = \tau_C\left(\sum_{n=1}^{N}(1-b_n(t))^+(\underline{\pi}_n(t) - \overline{\pi}_n(t))^+ - \sum_{n=1}^{N}(b_n(t)-1)^+\underline{\pi}_n(t) - \underline{l}(t)\right)^+. \tag{A.19}$$

Given that no resources are lost when portfolio rebalancing and paying taxes, equation (A.9) implies that the money market investment $\alpha(t)\,S_0(t)$ can be written proportional to $W(t)$ as

$$\alpha(t)\,S_0(t) = W(t)\left(1 - \sum_{n=1}^{N}\overline{\pi}_n(t) - c(t) - \phi^i_{CG}(t)\right), \quad i \in \{FUL, LUL\}, \tag{A.20}$$

where $c(t) \triangleq \frac{C(t)}{W(t)}$. Using (A.20), the margin constraint can also be written independent of wealth:

$$1 - c(t) - \phi^i_{CG}(t) \geq m_+\sum_{n=1}^{N}\overline{\pi}_n(t). \tag{A.21}$$

The wealth evolution equation (A.10) can also be written proportional to $W(t)$ implying

$$\frac{W(t+1)}{W(t)} = \frac{1}{1-\sum_{n=1}^{N}\underline{\pi}_n(t+1)(1+\delta_n(1-\tau_D))}\times \left[((1-\tau_D)\exp(r)+\tau_D)\left(1-\sum_{n=1}^{N}\overline{\pi}_n(t)-c(t)-\phi^i_{CG}(t)\right)\right], \quad i\in\{FUL, LUL\}. \tag{A.22}$$

Additionally, the stock proportion evolution and the accumulated capital loss evolution are given by

$$\underline{\pi}_n(t+1) = \frac{\frac{S_n(t+1)}{S_n(t)}\overline{\pi}_n(t)}{\frac{W(t+1)}{W(t)}}, \quad \underline{l}(t+1) = \frac{\bar{l}(t)}{\frac{W(t+1)}{W(t)}}, \tag{A.23}$$

where both quantities are independent of time t wealth. This evolution is needed in the dynamic programming formulation of the investor's problem. In particular, $\underline{\pi}_n$ is a state variable and $\overline{\pi}_n$ is a control variable.

Using the principle of dynamic programming and substituting out $W(t)$, the Bellman equation for the investor's optimization problem (A.13) in the FUL case is summarized by $2\times N+1$ state variables where we have two state variables for each stock and a state variable for time. After this change of variables, the Bellman equation is

$$V(t,\underline{\pi}(t),b(t)) = \max_{c(t),\overline{\pi}(t)} \frac{e^{-\lambda_t}c(t)^{1-\gamma}}{1-\gamma} + \frac{(1-e^{-\lambda_t})\beta(R^*)^{1-\gamma}}{(1-\beta)(1-\gamma)} + e^{-\lambda_t}\beta E_t\left[\left(\frac{e^{-i}W(t+1)}{W(t)}\right)^{(1-\gamma)} V(t+1,\underline{\pi}(t+1),b(t+1))\right], \tag{A.24}$$

for $t=0,1,\dots,T-1$ subject to the wealth evolution equation (A.22), the margin constraint (A.21), and the stock proportion dynamics (A.23). In the LUL case, an additional state variable is needed, $\underline{l}$, the accumulated capital losses. The Bellman equation for this investor's problem is given by

$$V(t,\underline{\pi}(t),b(t),\underline{l}(t)) = \max_{c(t),\overline{\pi}(t)} \frac{e^{-\lambda_t}c(t)^{1-\gamma}}{1-\gamma} + \frac{(1-e^{-\lambda_t})\beta(R^*)^{1-\gamma}}{(1-\beta)(1-\gamma)} + e^{-\lambda_t}\beta E_t\left[\left(\frac{e^{-i}W(t+1)}{W(t)}\right)^{(1-\gamma)} V(t+1,\underline{\pi}(t+1),b(t+1),\underline{l}(t+1))\right], \tag{A.25}$$

for $t=0,1,\dots,T-1$ subject to the wealth evolution equation (A.22), the margin constraint (A.21), and the stock/capital loss proportion dynamics (A.23). Note that $\underline{\pi}(t)$, $\overline{\pi}(t)$, and $b(t)$ are vectors of length N to capture the trading position and tax basis for each stock.

A.4 Liquidation Shocks

We now consider the portfolio problem outlined in Section 2 with liquidation shocks. Here, in addition to voluntary liquidation, the investor faces forced liquidation of embedded gains. Arrival of forced liquidation follows a Poisson process with rate Λ. When forced liquidation occurs, the investor liquidates at least a fraction, F, of his wealth. Given that a fraction, h, of wealth may be voluntarily liquidated, the investor is forced to liquidate an additional fraction, $(F-h)^+$, of wealth in positions with embedded gains, i.e., with $b<1$. When multiple assets have embedded gains, the asset with the lowest tax cost is liquidated first, i.e., the one with the highest relative tax basis. Next, the stock with the second highest relative tax basis is liquidated. To simplify, in the examples in this paper we assume that there is no withdrawal of funds upon liquidation.

For a Poisson process, the numbers of events counted in disjoint intervals are independent from each other and the number of events in time interval $(t, t+\tau]$ follows a Poisson distribution with associated parameter $\Lambda\tau$. As a result, the probability that at least one forced liquidation occurs in time interval $(t, t+1]$ for $t = 0, \ldots, T-1$, is calculated as

$$P(\Lambda) = 1 - \left.\frac{e^{-\Lambda}\Lambda^k}{k!}\right|_{k=0} = 1 - e^{-\Lambda} \tag{A.26}$$

which is a constant independent of time.

After rescaling, the value function, U, for the FUL case is calculated as

$$U(t, \underline{\pi}(t), b(t)) = [1 - P(\Lambda)]\,\tilde{V}(t, \underline{\pi}(t), b(t)) + P(\Lambda)\,\tilde{V}(t, \mathring{\underline{\pi}}(t), \mathring{b}(t)) \tag{A.27}$$

where $\mathring{\underline{\pi}}(t)$ and $\mathring{b}(t)$ are adjusted for forced liquidation and where

$$\begin{aligned}\tilde{V}(t, \underline{\pi}(t), b(t)) = \max_{c(t), \overline{\pi}(t)} \frac{e^{-\lambda_t} c(t)^{1-\gamma}}{1-\gamma} &+ \frac{(1 - e^{-\lambda_t})\beta(R^*)^{1-\gamma}}{(1-\beta)(1-\gamma)} \\ &+ e^{-\lambda_t}\beta E_t\left[\left(\frac{e^{-i}W(t+1)}{W(t)}\right)^{(1-\gamma)} U(t+1, \underline{\pi}(t+1), b(t+1))\right].\end{aligned} \tag{A.28}$$

In the LUL case, the value function is given by

$$U(t, \underline{\pi}(t), b(t), \underline{l}(t)) = [1 - P(\Lambda)]\,\tilde{V}(t, \underline{\pi}(t), b(t), \underline{l}(t)) + P(\Lambda)\,\tilde{V}(t, \mathring{\underline{\pi}}(t), \mathring{b}(t), \mathring{\underline{l}}(t)) \tag{A.29}$$

where $\mathring{\underline{\pi}}(t)$, $\mathring{b}(t)$ and $\mathring{\underline{l}}(t)$ are adjusted for forced liquidation and where

$$\begin{aligned}\tilde{V}(t, \underline{\pi}(t), b(t), \underline{l}(t)) = \max_{c(t), \overline{\pi}(t)} \frac{e^{-\lambda_t} c(t)^{1-\gamma}}{1-\gamma} &+ \frac{(1 - e^{-\lambda_t})\beta(R^*)^{1-\gamma}}{(1-\beta)(1-\gamma)} \\ &+ e^{-\lambda_t}\beta E_t\left[\left(\frac{e^{-i}W(t+1)}{W(t)}\right)^{(1-\gamma)} U(t+1, \underline{\pi}(t+1), b(t+1), \underline{l}(t+1))\right].\end{aligned} \tag{A.30}$$

Equations (A.28) and (A.30) share the same set of evolution rules and constraints as equations (A.24) and (A.25). Note that at the terminal time, $t = T$, the value function is $U = V$ and that no liquidation shock is expected at time 0. The value function is given by $U(0) = \tilde{V}(0)$.

B Numerical Optimization

To numerically solve the Bellman equations (A.24) and (A.25) (and equations (A.27) and (A.29)), we extend the methodology of Brandt et al. (2005) and Garlappi and Skoulakis (2008) to incorporate endogenous state variables and constraints on portfolio weights. In addition, since the state variable evolution is given by functions that are piecewise linear, the Bellman equation corresponds to a singular stochastic control problem that we solve employing a domain decomposition of the state space. We first briefly sketch the algorithm before providing additional details. A full description can be found in Yang (2010).[16]

[16]The parallel computing code used to solve the portfolio choice problems is available from the authors. As a run-time benchmark based on our computing resources, the two asset LUL portfolio choice problem takes approximately 90 hours to solve using 100 CPUs in parallel.

B.1 Sketch of Algorithm

Step 1 - Domain Decomposition

a. The state space is decomposed into degenerate and non-degenerate regions. The degenerate region corresponds to when a stock's basis-price ratio is above 1. The solution at a point in the degenerate region is mapped to a solution at a point in the non-degenerate region.

b. For a point in the non-degenerate region, the choice space is decomposed into partitions in such a way that, in each partition, the evolution of all state variables is differentiable (and linear).

Step 2 - Dynamic Programming

a. For each time step, starting at the terminal time and working backward, a quasi-random grid is constructed in the non-degenerate region of the state space. For each point on the grid, the value function, the optimal consumption, and the optimal portfolio decisions are computed.

b. The value function is approximated using a set of basis functions, consisting of radial basis functions and low order polynomials. This approximation is used in earlier time steps to compute conditional expectations of the value function.

Step 3 - Karush-Kuhn-Tucker (KKT) Conditions

To solve the Bellman equation for each point on the quasi-random grid in the non-degenerate region and for each partition in the choice space, the following steps are performed.

a. A Lagrangian function is constructed for the value function using the portfolio position constraints, the corresponding Lagrange multipliers, and the state variable evolution.

b. For each partition in the choice space, the system of first order conditions (KKT conditions) are constructed from the Lagrangian function.

c. The optimal solution of the KKT conditions is found using a double iterative process:

    i. An approximate optimal portfolio is chosen and the corresponding approximate optimal consumption is computed.

    ii. Given the approximate optimal consumption, the corresponding approximate optimal portfolio is updated by solving the system of KKT conditions. The solution is computed by approximating the conditional expectations in the derivatives of the Lagrangian function using a cross-test-solution regression:

        1. A quasi-random set of feasible allocations and consumptions is chosen.
        2. For each feasible choice, the required conditional expectations are computed using the approximate value function from the next time step that was already computed.
        3. For each feasible choice, the computed conditional expectations are projected on a set of basis functions of the choice variables. The basis functions are chosen such that the KKT system of equations is linear in the choice variables.
        4. The resulting linear system of equations is then solved.

    iii. The consumption choice is then updated to the choice corresponding to the new approximate optimal portfolio.

    iv. Step (ii) is repeated using a smaller region in which feasible portfolio choices are drawn. The region is chosen based on the location of the previously computed approximate optimal portfolio. This is the *test region contraction* step.

v. These steps are repeated until the consumption and portfolio choices converge.

We now provide a more detailed description of each step for the limited use of losses case. The full use of losses case is similar. As a reminder, the optimization problem being solved is equation (A.25):

$$
\begin{aligned}
V(t, \underline{\pi}(t), b(t), \underline{l}(t)) = \max_{c(t), \overline{\pi}(t)} & \frac{e^{-\lambda_t} c(t)^{1-\gamma}}{1-\gamma} + \frac{(1-e^{-\lambda_t})\beta (R^*)^{1-\gamma}}{(1-\beta)(1-\gamma)} \\
& + e^{-\lambda_t} \beta E_t \Bigg[\left(\frac{e^{-i} W(t+1)}{W(t)} \right)^{(1-\gamma)} V(t+1, \underline{\pi}(t+1), b(t+1), \underline{l}(t+1)) \Bigg],
\end{aligned}
$$

for $t = 0, 1, \ldots, T-1$ subject to the wealth evolution equation (A.22), the margin constraint (A.21), the stock/capital loss proportion dynamics (A.23), the basis-price evolution (A.15), the accumulated capital loss evolution (A.17), and the capital gain taxes (A.19).

B.2 Algorithm Step 1 - Domain Decomposition

The first step in solving the optimization problem is to decompose the state space into a degenerate and a non-degenerate region. The solution at any point in the degenerate region can be mapped to the solution at a point in the non-degenerate region, and the problem solved only over the non-degenerate region. The degeneracy arises when the basis-price ratio of a stock is above 1, in which case it is optimal to immediately liquidate the position and add the realized capital loss to the accumulated loss state variable.

Take as given a point in the state space $\left(\mathring{\underline{\pi}}(t) \in \mathbb{R}^N, \mathring{b}(t) \in \mathbb{R}_+^N, \mathring{l}(t) \in \mathbb{R}_+\right)$. We define the following sets:

the index set of all risky assets: $I = \{1, \ldots, N\}$,

the index set of degenerate assets: $I_t^D = \left\{i = 1, \ldots, N : \mathring{b}_i(t) > 1\right\}$,

the index set of non-degenerate assets: $I_t^{\bar{D}} = \left\{i = 1, \ldots, N : \mathring{b}_i(t) \leq 1\right\}$.

The set $\left(I_t^D, I_t^{\bar{D}}\right)$ forms a partition of I. Given any point $\left(\mathring{\underline{\pi}}(t), \mathring{b}(t), \mathring{l}(t)\right)$ in the state space, there exists an equivalent point $(\underline{\pi}(t), b(t), l(t))$ in the non-degenerate region of the state space, such that

$$
\begin{aligned}
V\left(t, \underline{\pi}(t), b(t), l(t)\right) &= V\left(t, \mathring{\underline{\pi}}(t), \mathring{b}(t), \mathring{l}(t)\right) \\
\overline{\pi}^*\left(t, \underline{\pi}(t), b(t), l(t)\right) &= \overline{\pi}^*\left(t, \mathring{\underline{\pi}}(t), \mathring{b}(t), \mathring{l}(t)\right) \\
c^*\left(t, \underline{\pi}(t), b(t), l(t)\right) &= c^*\left(t, \mathring{\underline{\pi}}(t), \mathring{b}(t), \mathring{l}(t)\right)
\end{aligned}
$$

where

$$
\underline{\pi}_i(t) = \begin{cases} 0 & \text{if } i \in I_t^D \\ \mathring{\underline{\pi}}_i(t) & \text{if } i \in I_t^{\bar{D}} \end{cases}, \quad b_i(t) = \begin{cases} 1 & \text{if } i \in I_t^D \\ \mathring{b}_i(t) & \text{if } i \in I_t^{\bar{D}} \end{cases}, \quad l(t) = \mathring{l}(t) + \sum_{i \in I_t^D} \left(\mathring{b}_i(t) - 1\right) \mathring{\underline{\pi}}_i(t).
$$

The second step employed in the domain decomposition is to decompose the choice space for each point in the non-degenerate region into partitions such that, in each partition, the piecewise linear constraints of the optimization problem become linear. This is achieved by choosing the following partitions:

Index set of stock positions when stock n's position reduced: $I_t^{RP} = \left\{ n \in I_t^{\bar{D}} : \overline{\pi}_n(t) \leq \underline{\pi}_n(t) \right\}$.

Index set of stock positions when stock n's position increased: $I_t^{IP} = \left\{ n \in I_t^{\bar{D}} : \overline{\pi}_n(t) > \underline{\pi}_n(t) \right\}$.

To find the optimal solution for each point in the non-degenerate part of the state space, we solve for each partition in the choice space and choose the solution with the higher value of the value function.

B.3 Algorithm Step 2 - Dynamic Programming

Given the structure of the non-degenerate region of the state space, and to ensure that we solve the optimization problem in a sufficiently dense set of points in the non-degenerate region, we further decompose the non-degenerate region into cases where assets are either held in non-zero, or in zero, amounts. The number of cases is equal to 2^N and the cases are enumerated below. In each region we generate a quasi-random grid on which we solve the optimization problem. The dimension of the grid in each region is twice the number of stocks which are held in non-zero positions, corresponding to the initial stock position and the basis-price ratio. An additional dimension is added to all grids, corresponding to the level of the carry-over loss.

	Asset 1	$\cdots$	Asset $N-1$	Asset N	Dimensions
Case 1	Long	$\cdots$	Long	Long	$2N$
Case 2	Long	$\cdots$	Long	Zero	$2\,(N-1)$
Case 3	Long	$\cdots$	Zero	Long	$2\,(N-1)$
Case 4	Long	$\cdots$	Zero	Zero	$2\,(N-2)$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
Case 2^N	Zero	$\cdots$	Zero	Zero	0

Once the optimal strategy and the value function levels are computed for all points in the quasi-random grid at a particular time, the value function for any point in the state space is approximated by projecting the values on a set of basis functions. Some form of approximation is necessary, since it is necessary to estimate the value function at arbitrary points in the state space in order to compute the conditional expectations that arise naturally when the optimization problem is solved at grid points in the previous time slice. In the literature different approximations have been used, including a linear rule (see Gallmeyer, Kaniel, and Tompaidis (2006)) and projection on polynomials of the state variables (see Brandt et al. (2005)). We choose an approximation scheme that proceeds in two steps. First, we project the value function on a set of low order polynomials of the state variables. Second, we approximate the residuals with a set of radial basis functions. Each radial basis function is defined by its weight, center, and width. We adjust the number of centers, the location of each center, the corresponding widths, and the corresponding weights to achieve a good approximation of the value function. Additional details of the radial basis function approximation are in Yang (2010).

B.4 Algorithm Step 3 - Karush-Kuhn-Tucker Conditions

To solve the optimization problem at each grid point in the non-degenerate region of the state space, we construct, as in Yang (2010), a Lagrangian function that combines the value function at time t with the constraints on the choice variables. The Lagrangian, given a point in the state space, is a function of the choice variables and the Lagrange multipliers.

To easily express the constraints (A.17) and (A.19), define the wealth-proportional realized capital gains or losses as

$$g(t) = \sum_{n=1}^{N}(1 - b_n(t))^+(\underline{\pi}_n(t) - \overline{\pi}_n(t))^+ - \sum_{n=1}^{N}(b_n(t) - 1)^+\underline{\pi}_n(t). \tag{B.1}$$

Then, equations (A.17) and (A.19) can be written as

$$\bar{l}(t) = \Big(\underline{l}(t) - g(t)\Big)^+, \quad \phi_{CG}^{LUL}(t) = \tau_C\Big(g(t) - \underline{l}(t)\Big)^+.$$

Since the terms $\Big(\underline{l}(t) - g(t)\Big)^+$ and $\Big(g(t) - \underline{l}(t)\Big)^+$ are non-differentiable when $\underline{l}(t) = g(t)$, it is necessary to write two versions of the Lagrangian and solve them separately depending on whether $g(t) \geq \underline{l}(t)$ or $g(t) \leq \underline{l}(t)$. Assuming $g(t) \geq \underline{l}(t)$, the Lagrangian at $(\underline{\pi}(t), b(t), \underline{l}(t))$ is

$$\begin{aligned}
\mathcal{L}\left(\overline{\pi}(t), c(t), \lambda_t^C, \lambda_t^m, \lambda_t^{RP}, \lambda_t^{IP}\right) &= e^{-\lambda_t}\frac{c(t)^{1-\gamma}}{1-\gamma} \\
&+ e^{-\lambda_t}\beta E_t\left[\left(\frac{e^{-i}W(t+1)}{W(t)}\right)^{1-\gamma} V\left(t+1, \underline{\pi}(t+1), b(t+1), \underline{l}(t+1)\right)\right] + \lambda_t^C(g(t) - \underline{l}(t)) \\
&+ \lambda_t^m\left[1 - c(t) - \phi_{CG}(t) - m^+\sum_{n=1}^{N}\overline{\pi}_n(t)\right] + \sum_{i\in I_t^{RP}}\lambda_t^{RP,i}\left[\underline{\pi}_i(t) - \overline{\pi}_i(t)\right] + \sum_{i\in I_t^{IP}}\lambda_t^{IP,i}\left[\overline{\pi}_i(t) - \underline{\pi}_i(t)\right],
\end{aligned}$$

where λ_t^C is the Lagrange multiplier corresponding to the constraint that the carry-over loss, after taxes are paid or returned, cannot be negative; λ_t^m is the Lagrange multiplier corresponding to the margin constraint; and $\lambda_t^{RP}, \lambda_t^{IP}$ are the Lagrange multipliers corresponding to the partitioning of the choice variable space.

Assuming $g(t) \leq \underline{l}(t)$, the Lagrangian at $(\underline{\pi}(t), b(t), \underline{l}(t))$ is

$$\begin{aligned}
\mathcal{L}\left(\overline{\pi}(t), c(t), \lambda_t^C, \lambda_t^m, \lambda_t^{RP}, \lambda_t^{IP}\right) &= e^{-\lambda_t}\frac{c(t)^{1-\gamma}}{1-\gamma} \\
&+ e^{-\lambda_t}\beta E_t\left[\left(\frac{e^{-i}W(t+1)}{W(t)}\right)^{1-\gamma} V\left(t+1, \underline{\pi}(t+1), b(t+1), \underline{l}(t+1)\right)\right] - \lambda_t^C(g(t) - \underline{l}(t)) \\
&+ \lambda_t^m\left[1 - c(t) - \phi_{CG}(t) - m^+\sum_{n=1}^{N}\overline{\pi}_n(t)\right] + \sum_{i\in I_t^{RP}}\lambda_t^{RP,i}\left[\underline{\pi}_i(t) - \overline{\pi}_i(t)\right] + \sum_{i\in I_t^{IP}}\lambda_t^{IP,i}\left[\overline{\pi}_i(t) - \underline{\pi}_i(t)\right],
\end{aligned}$$

where λ_t^C is the Lagrange multiplier corresponding to the constraint that the capital gain tax paid cannot be negative. All other Lagrange multipliers are the same as in the previous case.

The KKT conditions are derived by differentiating the Lagrangian with respect to the choice variables and Lagrange multipliers. The following conditional expectations of the value function at time $t+1$ are estimated:

$$\begin{aligned}
&E_t\left[\frac{\partial}{\partial\overline{\pi}_i(t)}\left[\left(\frac{W(t+1)}{W(t)}\right)^{1-\gamma} V\left(t+1, \underline{\pi}(t+1), b(t+1), \underline{l}(t+1)\right)\right]\middle|\, \underline{\pi}(t), b(t), \underline{l}(t), \overline{\pi}(t), c(t)\right], \\
&E_t\left[\frac{\partial}{\partial c(t)}\left[\left(\frac{W(t+1)}{W(t)}\right)^{1-\gamma} V\left(t+1, \underline{\pi}(t+1), b(t+1), \underline{l}(t+1)\right)\right]\middle|\, \underline{\pi}(t), b(t), \underline{l}(t), \overline{\pi}(t), c(t)\right].
\end{aligned}$$

To estimate the conditional expectations, for each point in the state space $(\underline{\pi}(t+1),\ b(t+1),\ \underline{l}(t+1))$, we generate a set of test values for the choice variables, $\left(\overline{\pi}^{(j)}(t), c^{(j)}(t)\right)_{j=1}^{n_t}$, and calculate the conditional expectation:

$$E_t\left[\left(\frac{W(t+1)}{W(t)}\right)^{1-\gamma} V\left(t+1,\underline{\pi}(t+1),b(t+1),\underline{l}(t+1)\right)\middle|\,\underline{\pi}(t),b(t),\underline{l}(t),\overline{\pi}^{(j)}(t),c^{(j)}(t)\right].$$

The test solutions need to be generated consistently with the partition of the choice space in which the problem is solved. Given the n_t values of the conditional expectation, we approximate, for each value of $(\underline{\pi}(t+1),b(t+1),\underline{l}(t+1))$, the conditional expectation at any value of the choice variables by projecting onto a set of basis functions $(f_k)_{k=1}^{n_b}$:

$$\begin{aligned}E_t\left[\left(\frac{W(t+1)}{W(t)}\right)^{1-\gamma} V\left(t+1,\underline{\pi}(t+1),b(t+1),\underline{l}(t+1)\right)\middle|\,\underline{\pi}(t),b(t),\underline{l}(t),\overline{\pi}^{(j)}(t),c^{(j)}(t)\right]\\ \approx\sum_{j=1}^{n_b}\omega_j\left(\underline{\pi}(t),b(t),\underline{l}(t)\right)f_j\left(\overline{\pi}(t),c(t)\right).\end{aligned}$$

We use basis functions f_k that are polynomials of the choice variables $(\overline{\pi}(t),c(t))$ up to order two. Once the conditional expectation is approximated, we approximate its derivatives by

$$\begin{aligned}E_t\left[\frac{\partial}{\partial\overline{\pi}_i(t)}\left[\left(\frac{W(t+1)}{W(t)}\right)^{1-\gamma} V\left(t+1,\underline{\pi}(t+1),b(t+1),\underline{l}(t+1)\right)\right]\middle|\,\underline{\pi}(t),b(t),\underline{l}(t),\overline{\pi}(t),c(t)\right]\\ \approx\sum_{j=1}^{n_b}\omega_j\left(\underline{\pi}(t),b(t),\underline{l}(t)\right)\frac{\partial}{\partial\overline{\pi}_i(t)}f_j\left(\overline{\pi}(t),c(t)\right),\\ E_t\left[\frac{\partial}{\partial c(t)}\left[\left(\frac{W(t+1)}{W(t)}\right)^{1-\gamma} V\left(t+1,\underline{\pi}(t+1),b(t+1),\underline{l}(t+1)\right)\right]\middle|\,\underline{\pi}(t),b(t),\underline{l}(t),\overline{\pi}(t),c(t)\right]\\ \approx\sum_{j=1}^{n_b}\omega_j\left(\underline{\pi}(t),b(t),\underline{l}(t)\right)\frac{\partial}{\partial c(t)}f_j\left(\overline{\pi}(t),c(t)\right).\end{aligned}$$

Given our choice of polynomials of order two, the KKT system of equations for each point in the non-degenerate part of the state space becomes a system of linear equations in terms of the optimal portfolios. To account for the inaccuracy in approximating conditional expectations with quadratic functions, we use an iterative scheme, where we successively reduce the size of the region from which the test solutions are drawn. Details of this procedure, termed the "Test Region Iterative Contraction (TRIC)," are provided in Yang (2010).

A final detail in solving the KKT system of equations, is that, given a guess for the optimal portfolio, $\overline{\pi}^a(t)$, we first solve for the optimal consumption by solving the equation

$$0=e^{-\lambda_t}c(t)^{-\gamma}+e^{-\lambda_t}e^{-i(1-\gamma)}\beta\sum_{j=1}^{n_b}\omega_j\left(\underline{\pi}(t),b(t),\underline{l}(t)\right)\frac{\partial}{\partial c(t)}f_j\left(\overline{\pi}^a(t),c(t)\right)-\lambda_t^m.$$

Once the approximate optimal consumption is calculated, we solve the remaining, linear, system of KKT equations for the optimal portofolio. This step involves solving the system of KKT equations in all the possible partitions of the choice space, and choosing the solution that maximizes the value function. In the next iteration, the region from which test solutions for the portfolio positions are drawn is contracted around the computed solution. The approximate portfolio is also used to update the approximation to the optimal consumption. The iteration is repeated until the difference between successive solutions is sufficiently small.

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Figure 2: **Example.** The figure reports properties of the optimal trading strategy for an LUL, an FUL, and an NCGT investor as a function of the investor's basis-to-price ratio, $b(0)$, at time $t = 0$, when the investor initially owns one share of stock and no bond position. The left panel summarizes after-tax optimal portfolio choices as a fraction of wealth $\overline{\pi}$ at $t = 0$ and $t = 1$ and wealth $W(0)$. The middle panel summarizes capital gain taxes paid Φ_{CG} at $t = 0$ and $t = 1$ as well as the investor's expected utility at $t = 0$. The right panel summarizes capital gain taxes paid at $t = 2$ when the investor consumes. 'Up' and 'Dn' denote up and down moves through the binomial tree. The values of the parameters used are given in Section 3.

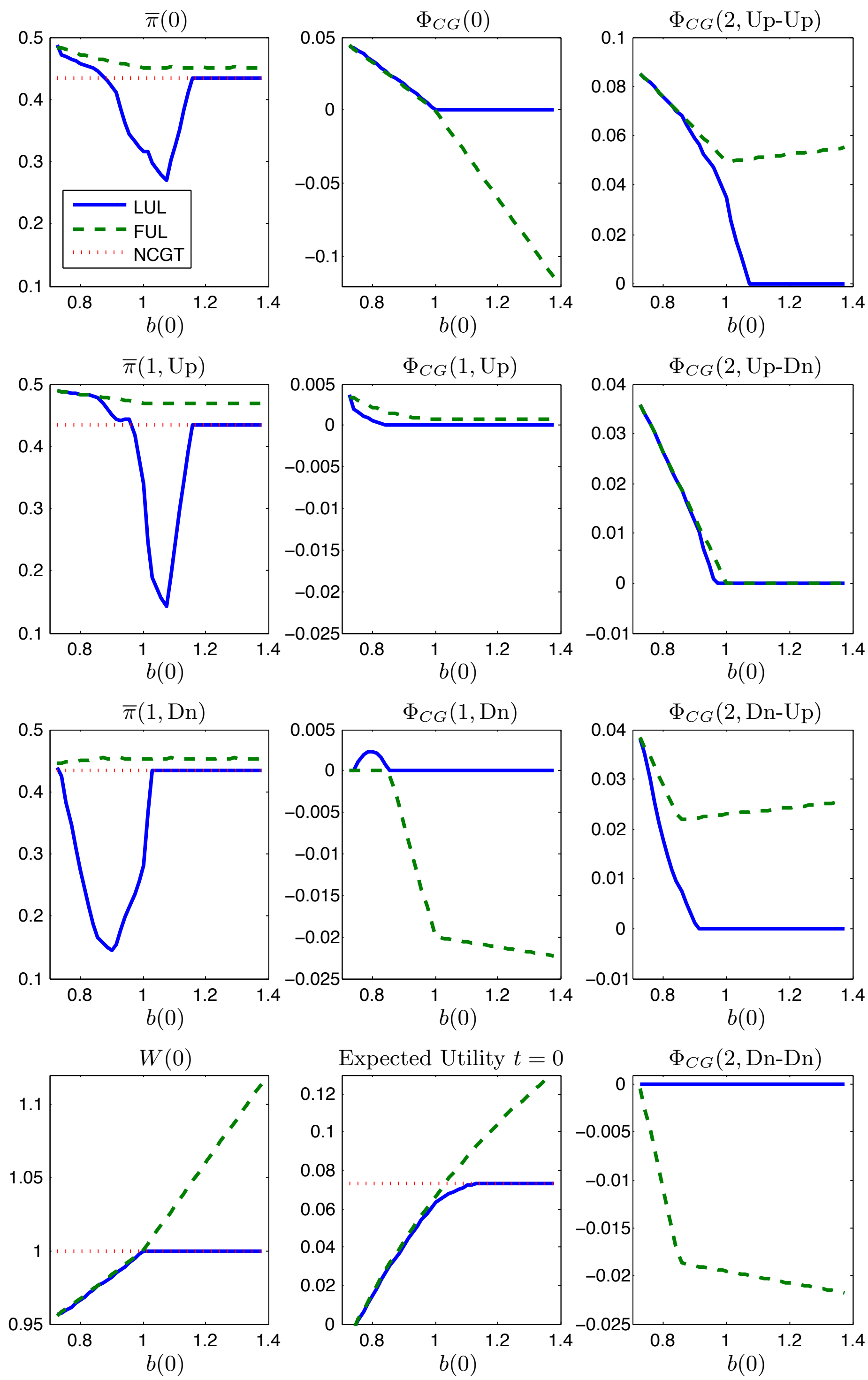

Figure 3: **Backtesting FUL versus LUL.** The figure reports backtests of the optimal trading strategy, $\overline{\pi}$, for an FUL and an LUL investor expressed as an equity-to-wealth ratio. Equity-to-wealth ratios are computed by dividing the value of the equity position by the value of the investor's wealth after subtracting consumption, and dividend and interest taxes, but before capital gains taxes. We find the investor's optimal FUL (solid line) and LUL (dashed line) strategy over their lifetime using the historical path for S&P 500 Index assuming that the backtest starts either in 1927 (top plots) or 1950 (middle plots) when the investor is 20 years old with initial stock position of 0, initial tax basis of 1, and initial carry-over loss of 0. The bottom plots show the equity-to-wealth ratio averaged by age. We find the optimal strategy for 55 investors that start in 1927, 1928, ..., 1981, respectively. The 55'th investor that enters the equity market in 1981 reaches age 50 in 2011. For the no liquidation shock (No LS) case, we assume no liquidation shock over the investor's lifetime from age 20 to age 99. For the LS case, we assume that the investor experiences a one-time 100% liquidation shock at age 50 (vertical black line) in the top and middle plots and at age 40 in the bottom plots. Parameters are set equal to their historical counterparts and are given in Section 6. Tax rates are set as in the Base Case (see Subsection 4.1).

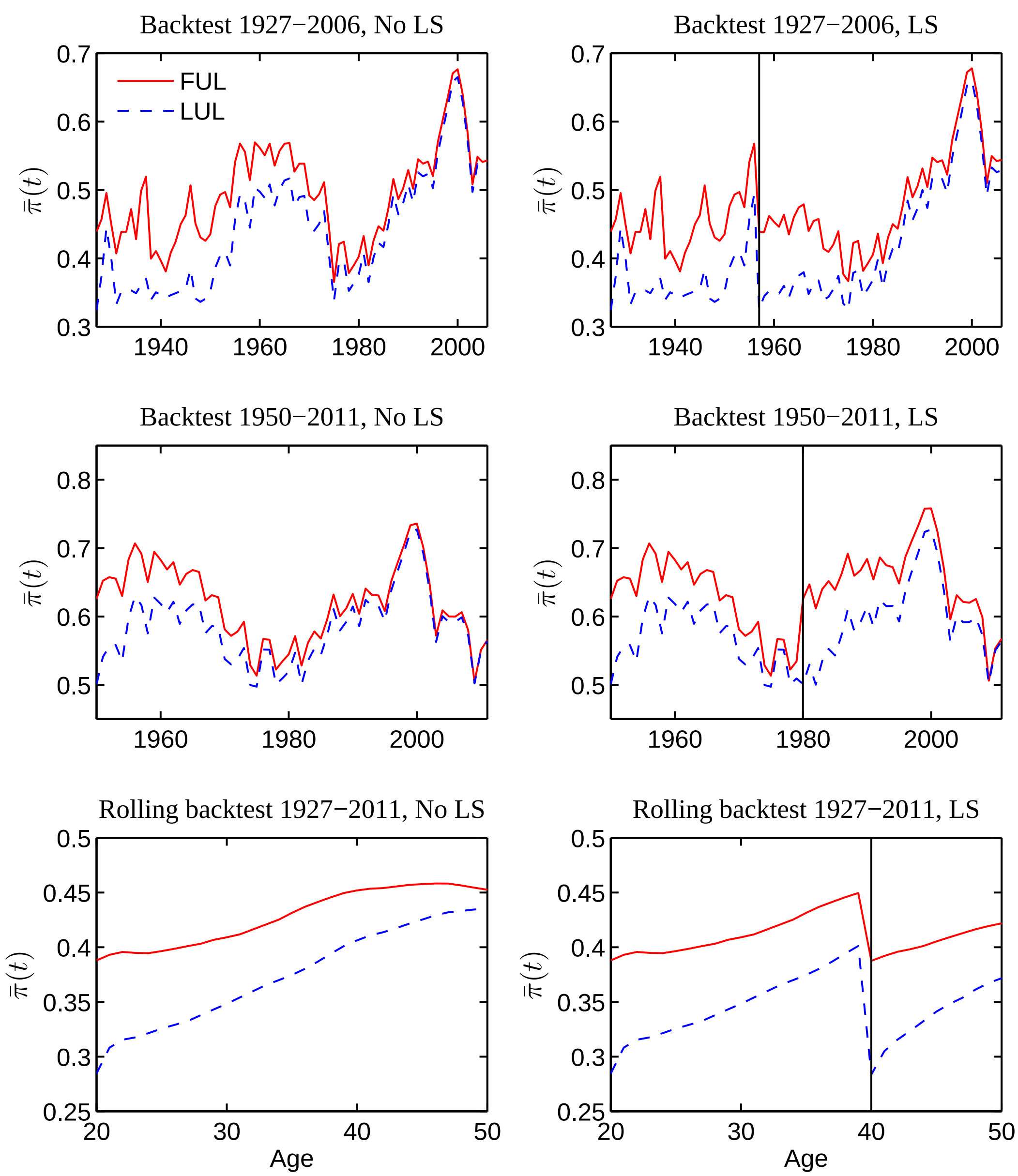

Table 1: **Base Case Optimal Strategies - One Stock.** The table summarizes optimal portfolio choice as a function of the equity-to-wealth ratio and the basis-to-price ratio entering the trading period for FUL portfolios and LUL portfolios for the Base Case. Equity-to-wealth ratios are computed by dividing the value of the equity position by the value of the investor's wealth after subtracting consumption, and dividend and interest taxes, but before capital gains taxes. In each panel, the equity-to-wealth ratios at age 20 and 80 are presented. The right panel computes the percentage increase in the FUL equity-to-wealth ratio relative to the LUL equity-to-wealth ratio. LUL optimal portfolios have a zero carry-over loss entering the trading period. Base Case parameters are summarized in Subsection 4.1.

Age 20

	FUL Exiting Equity-to-Wealth Ratio Entering Basis-to-Price Ratio								LUL Exiting Equity-to-Wealth Ratio Entering Basis-to-Price Ratio								% Change ((FUL-LUL)/LUL) Entering Basis-to-Price Ratio							
Entering Equity-to-Wealth Ratio	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
0.3	0.48	0.49	0.50	0.52	0.54	0.57	0.57	0.57	0.45	0.45	0.45	0.45	0.45	0.46	0.49	0.49	6.6%	7.6%	12.1%	15.3%	19.5%	23.3%	17.5%	16.3%
0.4	0.47	0.48	0.49	0.51	0.54	0.57	0.57	0.58	0.45	0.44	0.45	0.45	0.45	0.46	0.49	0.50	4.8%	7.1%	9.0%	13.9%	19.4%	24.1%	16.8%	16.1%
0.5	0.50	0.50	0.50	0.50	0.52	0.57	0.57	0.58	0.50	0.50	0.50	0.50	0.50	0.46	0.49	0.50	0.0%	0.0%	0.0%	0.0%	3.9%	24.0%	16.4%	16.3%
0.6	0.61	0.61	0.59	0.58	0.57	0.57	0.57	0.58	0.61	0.61	0.58	0.55	0.51	0.46	0.49	0.50	0.0%	0.0%	2.6%	4.9%	11.4%	24.1%	16.3%	16.6%
0.7	0.63	0.61	0.59	0.58	0.57	0.57	0.58	0.58	0.63	0.61	0.59	0.55	0.51	0.46	0.50	0.50	0.2%	0.8%	0.4%	5.5%	11.0%	24.0%	16.2%	16.9%

Age 80

Entering Equity-to-Wealth Ratio																								
0.3	0.50	0.51	0.53	0.54	0.56	0.58	0.58	0.59	0.47	0.47	0.47	0.47	0.47	0.47	0.49	0.50	5.8%	7.8%	12.8%	14.3%	19.3%	23.0%	18.1%	17.9%
0.4	0.48	0.49	0.51	0.53	0.56	0.58	0.59	0.59	0.47	0.47	0.47	0.47	0.47	0.47	0.50	0.50	2.4%	3.7%	9.5%	12.1%	17.9%	23.1%	18.0%	18.2%
0.5	0.50	0.50	0.50	0.51	0.55	0.58	0.59	0.59	0.50	0.50	0.50	0.50	0.50	0.47	0.50	0.50	0.0%	0.0%	0.1%	2.3%	8.9%	22.8%	18.0%	18.6%
0.6	0.60	0.60	0.60	0.60	0.59	0.58	0.59	0.59	0.60	0.60	0.60	0.59	0.54	0.47	0.50	0.50	0.0%	0.0%	0.0%	1.8%	9.6%	22.8%	17.9%	18.9%
0.7	0.70	0.68	0.64	0.62	0.59	0.58	0.59	0.60	0.70	0.67	0.63	0.59	0.54	0.47	0.50	0.50	0.0%	1.2%	1.6%	4.8%	9.7%	22.8%	18.1%	19.3%

Table 2: **Optimal Strategies - Two Stocks - Age 80.** The table summarizes optimal portfolio choice as a function of the basis-to-price ratios entering the trading period for four LUL cases: two stock Base Case, Capital Gain Tax 30% Case, Correlation 0.4 Case, and Correlation 0.9 Case. Equity-to-wealth ratios are computed by dividing the value of the equity position by the value of the investor's wealth after subtracting consumption, and dividend and interest taxes, but before capital gains taxes. All LUL optimal portfolios have a zero carry-over loss entering the trading period. The investor enters the period with a stock 1 (stock 2) position of $\underline{\pi}_1 = 0.3$ ($\underline{\pi}_2 = 0.4$). Two stock Base Case parameters are summarized in Subsection 4.1.

Age 80, 80% Correlation – LUL

	Equity-to-Wealth Ratio Stock 1								Equity-to-Wealth Ratio Stock 2								Total Equity-to-Wealth Ratio							
	Basis-to-Price Ratio Stock 2								Basis-to-Price Ratio Stock 2								Basis-to-Price Ratio Stock 2							
Basis-to-Price Ratio Stock 1	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
0.5	0.30	0.30	0.30	0.30	0.30	0.30	0.29	0.27	0.40	0.39	0.35	0.31	0.27	0.19	0.21	0.23	0.70	0.69	0.65	0.61	0.57	0.49	0.50	0.50
0.6	0.30	0.30	0.30	0.30	0.30	0.30	0.28	0.27	0.40	0.39	0.35	0.31	0.27	0.19	0.22	0.23	0.70	0.69	0.65	0.61	0.57	0.49	0.50	0.50
0.7	0.28	0.29	0.30	0.30	0.30	0.30	0.27	0.26	0.40	0.40	0.36	0.31	0.27	0.19	0.23	0.24	0.68	0.69	0.66	0.61	0.57	0.49	0.50	0.50
0.8	0.23	0.23	0.25	0.30	0.30	0.30	0.26	0.26	0.40	0.40	0.40	0.32	0.27	0.19	0.23	0.24	0.63	0.64	0.65	0.62	0.57	0.49	0.50	0.50
0.9	0.19	0.20	0.20	0.20	0.29	0.30	0.25	0.25	0.40	0.40	0.40	0.40	0.29	0.19	0.24	0.25	0.59	0.60	0.60	0.60	0.57	0.49	0.49	0.50
1.0	0.13	0.13	0.13	0.13	0.15	0.24	0.25	0.25	0.40	0.40	0.40	0.40	0.38	0.24	0.25	0.25	0.53	0.53	0.53	0.53	0.53	0.48	0.49	0.50
1.1	0.16	0.17	0.18	0.20	0.22	0.25	0.25	0.25	0.35	0.34	0.31	0.28	0.26	0.25	0.25	0.25	0.51	0.50	0.49	0.48	0.48	0.49	0.50	0.50
1.2	0.19	0.20	0.21	0.24	0.24	0.25	0.25	0.25	0.29	0.29	0.28	0.26	0.25	0.25	0.25	0.25	0.48	0.49	0.50	0.50	0.50	0.50	0.50	0.50

Age 80, 80% Correlation, 30% Tax Rate – LUL

Basis-to-Price Ratio Stock 1																								
0.5	0.30	0.30	0.30	0.30	0.30	0.30	0.29	0.27	0.40	0.40	0.40	0.37	0.29	0.19	0.21	0.23	0.70	0.70	0.70	0.68	0.60	0.49	0.50	0.50
0.6	0.30	0.30	0.30	0.30	0.30	0.30	0.28	0.27	0.40	0.40	0.40	0.37	0.29	0.19	0.22	0.23	0.70	0.70	0.70	0.68	0.60	0.49	0.50	0.50
0.7	0.30	0.30	0.30	0.30	0.30	0.30	0.27	0.26	0.40	0.40	0.40	0.38	0.30	0.19	0.22	0.24	0.70	0.70	0.70	0.68	0.60	0.49	0.50	0.50
0.8	0.29	0.30	0.30	0.30	0.30	0.30	0.26	0.26	0.40	0.40	0.40	0.39	0.30	0.19	0.23	0.24	0.70	0.70	0.70	0.69	0.60	0.49	0.50	0.50
0.9	0.22	0.21	0.22	0.23	0.30	0.30	0.25	0.25	0.40	0.40	0.40	0.40	0.31	0.19	0.24	0.25	0.62	0.62	0.62	0.63	0.61	0.49	0.49	0.50
1.0	0.13	0.13	0.13	0.13	0.13	0.24	0.25	0.25	0.40	0.40	0.40	0.40	0.40	0.24	0.25	0.25	0.53	0.53	0.53	0.53	0.54	0.48	0.49	0.50
1.1	0.16	0.17	0.18	0.19	0.21	0.25	0.25	0.25	0.35	0.34	0.31	0.28	0.27	0.25	0.25	0.25	0.51	0.50	0.49	0.48	0.48	0.49	0.50	0.50
1.2	0.19	0.19	0.21	0.24	0.24	0.25	0.25	0.25	0.29	0.30	0.28	0.26	0.25	0.25	0.25	0.25	0.48	0.49	0.50	0.50	0.50	0.50	0.50	0.50

Age 80, 40% Correlation – LUL

Basis-to-Price Ratio Stock 1																								
0.5	0.30	0.30	0.30	0.30	0.30	0.30	0.27	0.26	0.39	0.36	0.34	0.30	0.27	0.22	0.23	0.24	0.70	0.66	0.64	0.60	0.57	0.52	0.50	0.50
0.6	0.30	0.30	0.30	0.30	0.30	0.30	0.27	0.26	0.40	0.36	0.34	0.30	0.27	0.21	0.23	0.24	0.70	0.66	0.64	0.61	0.57	0.52	0.50	0.50
0.7	0.30	0.30	0.30	0.30	0.30	0.30	0.26	0.25	0.40	0.37	0.34	0.31	0.27	0.22	0.23	0.24	0.70	0.67	0.64	0.61	0.57	0.52	0.50	0.50
0.8	0.27	0.28	0.30	0.30	0.30	0.30	0.25	0.25	0.40	0.40	0.35	0.31	0.28	0.22	0.24	0.25	0.68	0.67	0.65	0.62	0.58	0.52	0.49	0.50
0.9	0.24	0.24	0.25	0.27	0.29	0.30	0.25	0.25	0.40	0.40	0.38	0.34	0.29	0.22	0.24	0.25	0.64	0.64	0.63	0.60	0.57	0.52	0.49	0.49
1.0	0.19	0.19	0.19	0.20	0.22	0.24	0.24	0.25	0.40	0.40	0.40	0.37	0.31	0.24	0.24	0.25	0.59	0.59	0.59	0.57	0.53	0.48	0.49	0.49
1.1	0.20	0.20	0.21	0.22	0.23	0.24	0.25	0.25	0.35	0.34	0.31	0.27	0.25	0.24	0.25	0.25	0.54	0.54	0.52	0.49	0.48	0.49	0.49	0.50
1.2	0.21	0.22	0.23	0.24	0.24	0.25	0.25	0.25	0.29	0.26	0.26	0.25	0.25	0.25	0.25	0.25	0.50	0.49	0.49	0.49	0.49	0.49	0.50	0.50

Age 80, 90% Correlation – LUL

Basis-to-Price Ratio Stock 1																								
0.5	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.28	0.40	0.40	0.35	0.31	0.27	0.19	0.20	0.22	0.70	0.70	0.65	0.61	0.57	0.49	0.50	0.50
0.6	0.30	0.30	0.30	0.30	0.30	0.30	0.29	0.28	0.40	0.40	0.35	0.31	0.27	0.19	0.21	0.23	0.70	0.70	0.66	0.61	0.57	0.49	0.50	0.50
0.7	0.28	0.28	0.30	0.30	0.30	0.30	0.28	0.27	0.40	0.40	0.37	0.31	0.27	0.19	0.21	0.23	0.68	0.68	0.67	0.61	0.57	0.49	0.50	0.50
0.8	0.21	0.21	0.22	0.30	0.30	0.30	0.27	0.26	0.40	0.40	0.40	0.32	0.27	0.19	0.22	0.24	0.61	0.61	0.62	0.62	0.57	0.49	0.50	0.50
0.9	0.19	0.18	0.19	0.19	0.29	0.30	0.26	0.26	0.40	0.40	0.40	0.40	0.28	0.19	0.23	0.24	0.59	0.59	0.59	0.59	0.57	0.49	0.49	0.50
1.0	0.10	0.10	0.10	0.10	0.10	0.24	0.25	0.25	0.40	0.40	0.40	0.40	0.40	0.24	0.25	0.25	0.50	0.50	0.50	0.50	0.50	0.48	0.49	0.50
1.1	0.12	0.15	0.17	0.19	0.20	0.25	0.25	0.25	0.37	0.33	0.31	0.29	0.28	0.25	0.25	0.25	0.49	0.49	0.49	0.48	0.48	0.49	0.50	0.50
1.2	0.17	0.18	0.20	0.23	0.24	0.25	0.25	0.25	0.33	0.32	0.29	0.27	0.26	0.25	0.25	0.25	0.50	0.50	0.49	0.50	0.50	0.50	0.50	0.50

Table 3: **Capital Gain Tax 30%, No Tax Forgiveness at Death, and Higher Risk Aversion - One Stock.** The table summarizes optimal portfolio choice as a function of the equity-to-wealth ratio and the basis-to-price ratio entering the trading period for three LUL cases: one stock Capital Gain Tax 30% Case, No Tax Forgiveness at Death Case, and Higher Risk Aversion Case. In each panel, the equity-to-wealth ratios at age 20 and 80 are presented. Equity-to-wealth ratios are computed by dividing the value of the equity position by the value of the investor's wealth after subtracting consumption, and dividend and interest taxes, but before capital gains taxes. All LUL optimal portfolios have a zero carry-over loss entering the trading period. Parameters used are summarized in Subsection 4.1.

Age 20

	Capital Gain Tax 30% - LUL Exiting Equity-to-Wealth Ratio Entering Basis-to-Price Ratio								No Tax Foregiveness at Death - LUL Exiting Equity-to-Wealth Ratio Entering Basis-to-Price Ratio									High Risk Aversion - LUL Exiting Equity-to-Wealth Ratio Entering Basis-to-Price Ratio							
Entering Equity-to-Wealth Ratio	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2		0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
0.3	0.45	0.45	0.45	0.45	0.45	0.45	0.48	0.49	0.47	0.48	0.47	0.47	0.47	0.48	0.49	0.50	0.05	0.22	0.22	0.22	0.22	0.23	0.23	0.23	0.23
0.4	0.45	0.44	0.44	0.45	0.45	0.45	0.49	0.50	0.47	0.47	0.48	0.47	0.47	0.47	0.50	0.50	0.15	0.21	0.21	0.22	0.22	0.22	0.23	0.24	0.24
0.5	0.50	0.50	0.50	0.50	0.50	0.45	0.49	0.50	0.50	0.50	0.50	0.50	0.50	0.47	0.50	0.50	0.25	0.25	0.25	0.25	0.25	0.25	0.23	0.24	0.25
0.6	0.61	0.61	0.61	0.58	0.53	0.45	0.49	0.50	0.61	0.60	0.58	0.56	0.52	0.47	0.50	0.50	0.35	0.33	0.32	0.31	0.28	0.26	0.23	0.24	0.25
0.7	0.68	0.64	0.61	0.57	0.53	0.45	0.49	0.50	0.61	0.60	0.58	0.56	0.52	0.47	0.50	0.50	0.45	0.33	0.32	0.30	0.28	0.26	0.23	0.25	0.25

Age 80

Entering Equity-to-Wealth Ratio																									
0.3	0.47	0.47	0.47	0.47	0.47	0.47	0.49	0.50	0.51	0.50	0.50	0.49	0.48	0.47	0.48	0.48	0.05	0.23	0.22	0.22	0.23	0.23	0.23	0.23	0.23
0.4	0.47	0.47	0.47	0.47	0.47	0.47	0.50	0.50	0.51	0.51	0.50	0.49	0.48	0.47	0.48	0.48	0.15	0.22	0.22	0.22	0.22	0.22	0.23	0.24	0.25
0.5	0.50	0.50	0.50	0.50	0.50	0.47	0.50	0.50	0.51	0.51	0.51	0.50	0.50	0.47	0.48	0.48	0.25	0.25	0.25	0.25	0.25	0.25	0.23	0.24	0.25
0.6	0.60	0.60	0.60	0.60	0.58	0.47	0.50	0.50	0.60	0.60	0.59	0.57	0.54	0.47	0.48	0.48	0.35	0.35	0.35	0.32	0.30	0.27	0.23	0.25	0.25
0.7	0.70	0.70	0.70	0.65	0.58	0.47	0.50	0.50	0.61	0.60	0.59	0.56	0.53	0.47	0.48	0.49	0.45	0.38	0.36	0.32	0.30	0.26	0.23	0.25	0.25

Table 4: **One Stock Simulations.** This table presents simulation results for LUL portfolio characteristics with (LS) and without liquidation shocks (no LS) at ages 40, 60, and 80 over 50,000 paths. Liquidation shock has 0.2 arrival rate and liquidation fraction is 1. The investor starts at age 20 with no embedded capital gains and zero carry-over loss. Equity-to-wealth ratios are computed by dividing the value of the equity position by the value of the investor's wealth after subtracting consumption, and dividend and interest taxes, but before capital gains taxes. One stock Base Case and Capital Gain Tax 30% Case parameters are summarized in Subsection 4.1.

One Stock Base Case - LUL

		Wealth ($)		Equity-to-Wealth Ratio		Basis-to-Price Ratio		Cumulative Capital Gain Tax-to-Wealth Ratio		Carry-Over Loss-to-Wealth Ratio	
	Percentile	No LS	LS	No LS	LS	No LS	LS	No LS	LS	No LS	LS
Age 40	5%	159.7	155.0	46.5%	49.5%	8.1%	28.9%	0.00%	0.84%	0.00%	0.00%
	10%	179.5	174.4	48.6%	49.6%	10.0%	39.6%	0.00%	1.98%	0.00%	0.00%
	25%	221.7	212.9	52.4%	50.2%	15.3%	60.7%	0.00%	4.23%	0.00%	0.00%
	50%	289.9	269.1	61.1%	51.5%	23.8%	79.7%	0.00%	7.25%	0.00%	0.00%
	75%	410.0	345.2	68.2%	55.1%	37.9%	100.0%	0.35%	10.37%	0.00%	4.52%
	90%	560.3	435.1	72.1%	61.0%	56.4%	100.0%	1.15%	12.93%	0.48%	12.17%
	95%	665.1	505.8	73.1%	63.0%	69.8%	100.0%	1.62%	14.24%	5.08%	17.89%
	Mean	336.8	291.7	60.5%	53.3%	29.3%	76.2%	0.30%	7.38%	0.83%	3.64%
	Std. Dev.	171.2	113.9	8.7%	4.3%	19.3%	24.1%	0.57%	4.08%	3.70%	7.00%
Age 60	5%	359.2	335.7	48.5%	48.6%	1.2%	28.0%	0.00%	4.06%	0.00%	0.00%
	10%	433.3	398.4	51.8%	48.6%	1.7%	38.7%	0.00%	5.34%	0.00%	0.00%
	25%	621.7	534.0	59.6%	49.3%	3.2%	59.5%	0.00%	7.98%	0.00%	0.00%
	50%	993.0	744.1	68.0%	50.7%	6.2%	79.7%	0.22%	10.99%	0.00%	0.00%
	75%	1,654.0	1,051.7	74.1%	54.7%	12.5%	100.0%	0.94%	13.77%	0.00%	4.42%
	90%	2,593.2	1,449.0	77.5%	61.0%	24.2%	100.0%	1.60%	15.90%	0.00%	11.53%
	95%	3,512.8	1,778.2	78.2%	64.2%	36.0%	100.0%	1.95%	17.01%	0.00%	17.09%
	Mean	1,341.2	862.5	66.3%	52.8%	10.6%	75.7%	0.54%	10.81%	0.10%	3.47%
	Std. Dev.	1,181.3	496.4	9.3%	4.9%	12.6%	24.4%	0.69%	3.96%	1.37%	6.80%
Age 80	5%	908.4	781.9	50.8%	47.9%	0.2%	27.9%	0.00%	5.02%	0.00%	0.00%
	10%	1,186.5	973.3	54.9%	47.9%	0.3%	38.7%	0.00%	6.55%	0.00%	0.00%
	25%	1,952.9	1,391.1	63.0%	48.4%	0.7%	59.6%	0.00%	9.31%	0.00%	0.00%
	50%	3,618.3	2,084.6	71.4%	50.0%	1.7%	79.7%	0.16%	12.25%	0.00%	0.00%
	75%	7,032.2	3,192.3	77.9%	53.8%	4.1%	100.0%	0.49%	14.74%	0.00%	4.32%
	90%	12,791.0	4,738.1	82.1%	60.6%	10.7%	100.0%	0.84%	16.60%	0.00%	11.28%
	95%	18,326.0	6,072.9	84.1%	65.1%	19.2%	100.0%	1.07%	17.60%	0.00%	16.52%
	Mean	5,939.8	2,595.8	69.9%	52.2%	4.4%	75.7%	0.30%	11.91%	0.01%	3.39%
	Std. Dev.	7,833.5	1,934.3	10.1%	5.6%	8.3%	24.4%	0.38%	3.80%	0.46%	6.59%

One Stock Capital Gain Tax 30% Case - LUL

Age 40	5%	159.8	152.4	46.4%	50.0%	8.1%	28.9%	0.00%	0.94%	0.00%	0.00%
	10%	179.6	170.4	48.5%	50.2%	10.0%	39.7%	0.00%	2.66%	0.00%	0.00%
	25%	221.4	205.9	52.4%	50.2%	15.3%	60.9%	0.00%	6.34%	0.00%	0.00%
	50%	289.1	257.4	61.3%	52.4%	23.8%	79.7%	0.00%	11.33%	0.00%	0.00%
	75%	408.6	327.7	70.1%	56.3%	37.9%	100.0%	0.00%	16.47%	0.00%	4.58%
	90%	569.6	413.5	76.6%	62.8%	56.5%	100.0%	0.00%	20.79%	0.40%	12.31%
	95%	696.5	480.5	79.6%	66.7%	69.8%	100.0%	0.13%	23.09%	5.02%	18.05%
	Mean	341.5	279.7	61.8%	54.3%	29.3%	76.2%	0.04%	11.61%	0.82%	3.74%
	Std. Dev.	187.2	108.7	10.4%	5.3%	19.3%	24.1%	0.23%	6.75%	3.69%	7.05%
Age 60	5%	359.0	319.7	49.0%	48.5%	1.2%	28.0%	0.00%	5.78%	0.00%	0.00%
	10%	433.0	374.2	52.4%	48.7%	1.7%	38.7%	0.00%	8.05%	0.00%	0.00%
	25%	619.6	493.0	60.9%	48.8%	3.2%	59.8%	0.00%	12.57%	0.00%	0.00%
	50%	998.9	675.5	70.4%	50.8%	6.2%	79.7%	0.00%	17.61%	0.00%	0.00%
	75%	1,724.7	945.7	78.0%	55.4%	12.3%	100.0%	0.00%	22.26%	0.00%	4.43%
	90%	2,904.0	1,297.7	82.8%	62.1%	24.0%	100.0%	0.15%	25.94%	0.00%	11.47%
	95%	4,105.7	1,589.0	85.0%	66.4%	35.8%	100.0%	0.49%	27.91%	0.00%	17.04%
	Mean	1,457.6	781.5	69.0%	53.3%	10.5%	75.8%	0.06%	17.33%	0.10%	3.54%
	Std. Dev.	1,528.3	443.4	11.1%	5.8%	12.6%	24.4%	0.23%	6.71%	1.36%	6.78%
Age 80	5%	906.6	713.7	51.6%	47.2%	0.2%	28.0%	0.00%	7.79%	0.00%	0.00%
	10%	1,183.7	874.8	56.1%	47.3%	0.3%	39.0%	0.00%	10.44%	0.00%	0.00%
	25%	1,955.0	1,218.3	64.8%	47.4%	0.7%	59.9%	0.00%	14.95%	0.00%	0.00%
	50%	3,712.5	1,782.5	73.5%	49.7%	1.6%	79.7%	0.00%	19.79%	0.00%	0.00%
	75%	7,681.7	2,683.9	80.2%	54.1%	3.9%	100.0%	0.00%	23.93%	0.00%	4.30%
	90%	15,090.0	3,949.2	84.5%	60.7%	10.0%	100.0%	0.04%	27.12%	0.00%	11.05%
	95%	22,813.0	5,056.1	86.5%	65.5%	18.3%	100.0%	0.13%	28.85%	0.00%	16.19%
	Mean	6,955.4	2,205.8	71.9%	52.1%	4.3%	75.7%	0.02%	19.26%	0.01%	3.43%
	Std. Dev.	11,654.6	1,608.0	10.6%	6.1%	8.1%	24.3%	0.08%	6.36%	0.46%	6.49%

Table 5: **Economic Cost of Taxation.** This table reports wealth equivalent changes in percent of an age 20 NCGT investor due to imposing a capital gain tax. The investor is assumed to initially have no embedded capital gains or losses in his portfolio. Wealth equivalent changes are computed such that the investor's utility is the same from the NCGT case to the corresponding capital gain tax case. A positive percentage wealth equivalent change denotes the NCGT investor's welfare improves by paying a capital gain tax. Results are reported for five LUL and FUL cases: no liquidation shock; liquidation shock with 0.2 arrival rate and liquidation fraction 0.5; liquidation shock with 0.2 arrival rate and liquidation fraction 0.75; liquidation shock with 0.2 arrival rate and liquidation fraction 1; and liquidation shock arrival every period and liquidation fraction 1. The top Panel reports results for one stock Base Case, while the bottom Panel reports results for the Capital Gain Tax 30% Case. One stock Base Case and Capital Gain Tax 30% Case parameters are summarized in Subsection 4.1.

One Stock Base Case

Liquidation shock	No	Yes	Yes	Yes	Yes
Liquidation arrival rate	—	0.2	0.2	0.2	Every period
Liquidation fraction	—	0.5	0.75	1	1
Wealth Equivalent - FUL	2.16%	1.53%	-1.13%	-1.94%	-4.41%
Wealth Equivalent - LUL	-0.45%	-0.84%	-3.98%	-5.69%	-8.48%

One Stock Capital Gain Tax 30% Case

Wealth Equivalent - FUL	3.64%	2.24%	-2.10%	-3.41%	-7.50%
Wealth Equivalent - LUL	-0.47%	-1.20%	-6.29%	-9.08%	-13.49%