

Leverage and Credit Risk Dynamics under Costly Equity Issuance*

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Abstract

We develop a dynamic model of leverage and credit risk pricing, featuring downward jump cash-flow shocks and costly external equity. The firm begins with a low target leverage and rolls over its debt at equilibrium interest rates, allowing leverage to drift in response to realized earnings. As leverage rises, the firm proactively deleverages through recapitalization, preserving financial slack and mitigating default risk. Default occurs unpredictably following a downward jump shock and becomes more likely as leverage increases, causing equilibrium credit spreads to rise with leverage. The firm is endogenously risk-averse at low leverage but becomes risk-seeking at high leverage.

Keywords: jump shocks, credit spreads, costly external equity, debt capacity, tradeoff theory, pecking order, financial flexibility, risk seeking

JEL Codes: G32, G33, G35, E22

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1 Introduction

“How did you go bankrupt? Two ways. Gradually, then suddenly.”

Ernest Hemingway, The Sun Also Rises.

In his AFA presidential address DeMarzo (2019) states that “Capital structure is not static, but rather evolves over time as an aggregation of sequential decisions.” New decisions are called for in response to realized profits and losses, or changes in the firm’s investment opportunity set. We take such a dynamic perspective and propose a tractable model that accounts for CFOs’ stated top four concerns, as reported in Graham and Harvey (2001) and Graham (2022): 1.) maintain financial flexibility, 2.) target credit ratings, 3.) preserve internal funds, and 4.) reduce earnings volatility.

We study these considerations in a dynamic tradeoff model: debt offers tax, informational and other benefits relative to equity, as in the classic tradeoff theory (Miller and Modigliani, 1961; Miller, 1977; Myers, 1984; Leland, 1994a), while external equity is costly, both because of adverse-selection–induced dilution (Myers and Majluf, 1984) and direct transaction and flotation costs. These costs give debt a near-term funding advantage over equity but can lead to excessively high leverage over time, prompting firms to proactively deleverage by issuing equity at a cost to postpone an even more costly default.

To capture the resulting rich state-contingent tradeoffs, we develop a model of the intertwined dynamics of leverage and credit risk pricing under a jump-diffusion process for corporate earnings before interest and taxes (EBIT). Jump risk is essential: only with jumps does the firm face sudden bankruptcy, consistent with the evidence. Anticipating the firm’s default strategy, credit markets price its debt with an equilibrium credit spread, and this dynamic interaction between the firm and its creditors continues until the firm defaults.

Costly external equity is also essential. Absent it, the model would counterfactually predict a constant target leverage ratio, excessively frequent equity issuance and payouts, and a time-invariant credit spread. With costly external equity, the firm’s response to a downward cash-flow jump depends on both its size and the firm’s current leverage: the firm absorbs a small jump by borrowing more, undertakes a costly recapitalization by issuing equity for a medium-size jump,

or defaults following a large one. What counts as small, medium, or large depends on current leverage. Taking these state-contingent responses into account, we turn to the equilibrium pricing of debt and equity.

As in Black and Scholes (1973) and Abel (2018), debt in our model is short term, maturing instantaneously, so that all outstanding debt is repaid before any new debt is issued. Default can then only be triggered by sufficiently large downward jumps. This key feature allows us to characterize the equilibrium credit spread via a closed-form integral rather than by a differential equation. Using this integral expression, we obtain a tractable integro-differential equation for the firm's equity value, substantially simplifying our joint analysis of leverage and pricing. Unlike in structural credit-risk models with term debt à la Leland (1994a), diffusion shocks alone cannot trigger default in our model, because the firm continuously adjusts its liabilities and can issue external equity if necessary.

We can interpret this instantaneous debt as a reduced-form proxy for total corporate debt rather than as just a specific instrument. This is because the instantaneous debt policy can be implemented through a long-term lending contract under which a single lender commits to a state-contingent debt and interest-rate schedule. This long-term contract with a single lender is in the spirit of the covenant-based margin pricing (credit spread) grids used in real-world syndicated loans.¹ We develop this equivalence duality result in subsection 4.6.

Depending on its history of past earnings shocks and financing decisions, the firm may find itself in one of four regions for the leverage state variable (see Figure 1): from left to right, 1.) a payout region, where the firm pays a lumpy dividend and reaches an optimal target leverage; 2.) a target zone, bordered by the target leverage and the proactive deleveraging threshold, in which debt and leverage evolve in response to profits and losses; 3.) an equity-issuance zone, bordered by the deleveraging threshold and the firm's debt capacity, in which the firm proactively recapitalizes to a recapitalization target; and 4.) a default region, beyond the debt capacity, where the firm optimally defaults.

These four regions are connected via market leverage dynamics (ΔML) defined as follows:

$$\Delta ML = \text{equity payout} - \text{retained earnings} - \text{equity issuance} + (100\% - ML),$$

¹ See, e.g., Asquith, Beatty, and Weber (2005) and Ivashina and Kovner (2011).

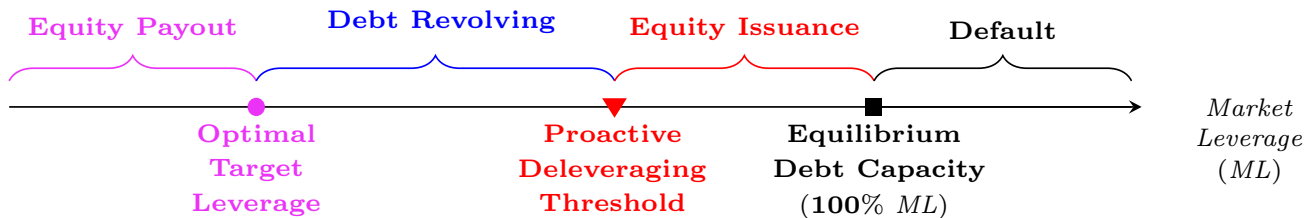


Figure 1: FOUR-REGION SOLUTION FOR MARKET LEVERAGE (ML). In the equity payout region, the firm makes a lumpy payout to reach optimal target leverage. In the debt revolving region, the firm retains earnings and rolls over its debt. In the equity issuance region, the firm proactively deleverages. In the default region, the firm's ML would have exceeded one without default. The three thresholds, target leverage, proactive deleveraging threshold, and debt capacity, are endogenously determined.

where each term indicates the key margin the firm uses to manage its leverage and corresponds to the color-matched region in Figure 1. This figure also reveals the firm's preferred financing order: first retained earnings, second debt, third equity — a pecking-order flavor (Myers, 1984), but with two key differences. First, the firm proactively issues costly equity to preempt future default risk. Second, the firm reverts to a recapitalization target that generally differs from the optimal target leverage, since issuing external equity is costly.

Figure 2 depicts the resulting nonlinear, non-monotonic leverage dynamics with four coloured arrows. The red arrow describes equity-financed deleveraging to the recapitalization target (the blue star $\star$). The two blue arrows describe the dynamics within the target zone: leverage drifts toward the payout boundary at lower leverage and away from it in the debt-spiral region. In this region, higher leverage raises the equilibrium credit spread, increasing interest expenses and borrowing needs; additional borrowing can, in turn, further raise leverage and credit spreads. Costly external equity makes the firm willing to remain in this region until recapitalization becomes optimal. This feedback links financing decisions to credit-risk pricing. If leverage is too low, the firm issues debt and makes a payout to reach target leverage (magenta arrow); if leverage exceeds debt capacity, it defaults.

Due to the interaction between jumps and fixed equity issuance costs, firm value is concave for low values of leverage, so that the firm is then locally endogenously risk-averse, and convex for high values of leverage, so that the firm is then endogenously risk-seeking. This is a prediction that is absent in other dynamic models, e.g., Hennessy and Whited (2007) and Bolton, Chen, and Wang (2011).

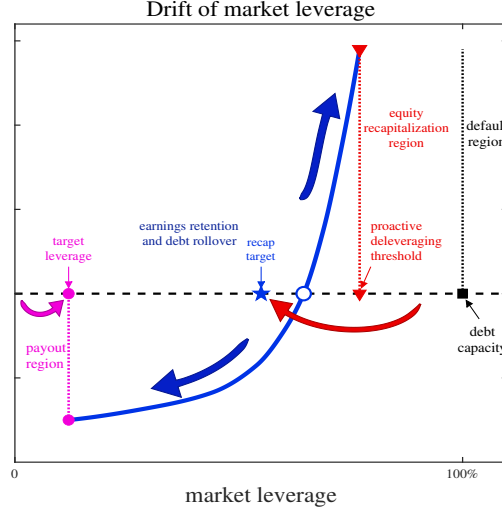


Figure 2: Graphical illustration of market leverage dynamics.

Our quantitative analysis (Section 5) generates deleveraging and credit-risk dynamics that are widely consistent with empirical findings. First, our model generates outcomes broadly consistent with the empirical estimates reported by DeAngelo, Gonçalves, and Stulz (2018) for median market leverage at the peak and subsequent trough, the median peak-to-trough decline, and the duration of deleveraging. Our model also predicts that credit spreads are narrow at low leverage but rise sharply and nonlinearly as leverage increases, broadly matching the one-year CDS spreads reported by Huang, Shi, and Zhou (2020). In addition, the model-implied default probabilities align with the estimates of Huang and Huang (2012), and the model-implied equity issuance frequency matches the estimated probability of seasoned equity offerings (SEOs) in DeAngelo, DeAngelo, and Stulz (2010).

Our model implies an average market leverage of 23%, within the 20%-25% range reported by Strebulaev and Whited (2012), even with a substantial tax advantage of debt. This result is driven by the interaction between downward earnings jumps and costly external equity. Paradoxically, higher equity issuance costs lower the firm's average leverage: even a small fixed cost widens the target zone and lowers average leverage. We also show that average leverage is sensitive to cash-flow risk, especially jump risk, but only moderately sensitive to the corporate tax rate and largely insensitive to the liquidation recovery value. In sum, our model explains the prudent leverage management observed for listed companies (Graham and Harvey, 2001).

We extend the baseline model in three directions. Section 6 shows that the firm may also save, when operating cash flow is subject to a second, contemporaneous shock, meaning that the firm optimally borrows in some states and holds cash in others. Appendix I extends the baseline model to let the firm hedge and/or seek additional risk exposure by trading a risky asset. Appendix J generalizes the baseline model to allow for growth options and capital accumulation.

Related literature. Our paper is closely related to Abel (2018), who was the first to develop a dynamic tradeoff model with short-term debt. Our model retains this debt structure but differs from his in two respects: costly external equity and the specification of the EBIT process. To distinguish their effects on leverage dynamics, we first consider a costless-equity benchmark of our model in Section 3 and then introduce equity issuance costs into this benchmark.

The first comparison highlights the role of the EBIT specification. In Abel (2018), external equity is costless and cash flow remains constant between Poisson arrivals, at which it is redrawn from a time-invariant distribution. His optimal policy is characterized by a single threshold: debt remains constant in levels when EBIT is above an endogenous threshold, and the firm is fully levered below it (i.e., 100% leverage). Under this policy, the firm never voluntarily reduces its outstanding debt. Our costless-equity benchmark instead implies a constant target debt-EBIT ratio because EBIT growth in our model is independent and identically distributed (i.i.d.), with downward jumps. The second comparison isolates the effect of equity issuance costs within our model. These costs generate a leverage target zone and discrete equity-financed deleveraging, replacing the prediction of a constant target leverage ratio. The interaction between jumps and fixed equity issuance costs also generates an endogenous risk attitude—risk-averse at low leverage and risk-seeking at high leverage—that is absent from both costless-equity benchmarks.

Our model is related to the contingent-claim leverage-dynamics and credit-risk literature initiated by Leland (1994a), which also assumes costless equity issuance but typically works with term debt.² Building on Leland (1994a) and allowing the firm to continuously adjust its debt, DeMarzo and He (2021) show that, without commitment to future debt issuance, shareholders

² In Leland (1994a,b), leverage dynamics are driven by an exogenous stochastic firm-value process as the firm pre-commits to a debt structure; Goldstein, Ju, and Leland (2001) work with a fixed rollover rule (also see Strebulaev, 2007).

gradually issue debt and never voluntarily repurchase outstanding debt. Their incentives to dilute existing debt dissipate all value gains from adjusting leverage in equilibrium.³

The contrast is therefore in the incentives underlying debt adjustment: following a sufficiently adverse earnings jump, our firm may issue costly equity to reduce debt and manage default risk, whereas shareholders in DeMarzo and He (2021) have incentives to issue additional debt and never voluntarily repurchase outstanding debt.

In a related model with downward EBIT jumps and Markov perfect equilibrium punishment strategies should the firm deviate from the announced debt policy, Malenko and Tsoy (2025) show that the optimal time-consistent debt policy features two regimes: a stable regime and a distress regime. The optimal time-consistent debt strategy requires the firm to (1.) continuously and actively manage its liabilities in the stable regime so that its debt-EBIT ratio stays at the optimal (constant) target and (2.) passively allow its leverage to evolve in response to EBIT shocks in the distress regime, into which the firm may transition after being hit by downward jump shocks.

Benzoni et al. (2022) show that fixed debt issuance costs mitigate the ratcheting incentive in the DeMarzo-He setting, and Hu, Varas, and Ying (2026) study a setting with regime-switching cash-flow growth and both short- and long-term debt, highlighting the tradeoff between the commitment value of short-term debt and the hedging benefits of long-term debt.

Our main contribution to the leverage-dynamics literature is to show that costly external equity makes a firm’s leverage, earnings retention, equity issuance, payout, and default interdependent, resulting in a four-region, highly nonlinear and nonmonotonic leverage policy featuring a wide target zone and proactive equity recapitalization.

On credit-risk pricing specifically, this literature splits between the structural approach following Merton (1974) and Leland (1994a) and the jump-intensity reduced-form approach following Duffie and Singleton (2003); the former is well grounded in dynamic optimization but tends to predict spreads that are too low, especially near maturity, while the latter does not price default endogenously. We combine the strengths of both approaches. As in Duffie and Singleton (2003), default is doubly stochastic, meaning that both the timing and likelihood of default

³ This debt-ratcheting incentive does not imply that the debt balance or leverage can only rise: contractual debt retirement (at an exogenously specified rate) can reduce the debt balance, and leverage thus can mean-revert through asset growth and debt maturity.

are random, generating predictions consistent with evidence, e.g., positive credit spreads even for bonds with very short maturity. At the same time, equilibrium credit risk results from the dynamic interaction between an optimizing firm and creditors who break even in bond markets. Despite this endogeneity, we obtain an essentially closed-form, highly nonlinear credit spread.

Our theory is also related to the continuous-time corporate liquidity management literature.⁴ That literature typically treats debt capacity and the borrowing cost as exogenous (Bolton et al., 2011, e.g.); in our model, debt plays both a liquidity and a leverage role, and leverage, the equilibrium credit spread, and debt capacity are all jointly determined.

Our theory of leverage dynamics is also related to the discrete-time dynamic corporate finance literature, see, e.g., Hennessy and Whited (2005, 2007).⁵ Continuous time allows us to cleanly separate diffusion shocks from discrete jump shocks, so that we can clearly see that jumps alone drive default. This feature yields a two-step solution procedure in which we first solve the credit-spread integral and then the equity-value integro-differential equation. This procedure is both more tractable and more transparent than the “loop-within-a-loop” iteration required in discrete time, where the credit spread and equity value must be solved jointly. See Strebulaev and Whited (2012) for a discussion of numerical complexity in the discrete-time literature. Separating jump and diffusion risks also yields results not present in Hennessy and Whited (2007) and makes the intuition behind some of their findings more transparent.

The remainder of this paper is organized as follows. Section 2 introduces the model with costly external equity. Section 3 solves the special case of our model with costless external equity, serving as a benchmark against which we assess the effect of costly external equity. Section 4 solves the model with costly external equity and characterizes the four-region policy, including the equivalent long-term-contract implementation in subsection 4.6. Section 5 analyzes empirical predictions. Section 6 extends the model to allow the firm to also save. Section 7 concludes.

⁴ See Bolton, Chen, and Wang (2011), Décamps, Mariotti, Rochet, and Villeneuve (2011), Hugonnier, Malamud, and Morellec (2015), Décamps et al. (2016), Della Seta, Morellec, and Zucchi (2020), Gryglewicz et al. (2022), Abel and Panageas (2023), and Hartman-Glaser, Mayer, and Milbradt (2025), for example.

⁵ Important early work on dynamic investment under costly external financing includes Cooley and Quadrini (2001) and Gomes (2001). Moyen (2004) also features equilibrium default and credit-risk pricing, but debt is risk free in the related Hennessy and Whited (2005), so that paper has no credit-risk implications. Riddick and Whited (2009) focus on corporate savings, with no credit-risk analysis. Hennessy and Whited (2007)’s focus is to infer financing costs via simulated method of moments, while we develop a tractable model to generate new testable quantitative and empirical predictions.

2 Baseline Model

Investors are risk averse and demand a risk premium for bearing systematic risk. We thus specify the firm's EBIT process on a risk-adjusted basis, that is, under the risk-neutral measure $\mathbb{Q}$. Working directly under $\mathbb{Q}$ is for ease of exposition, allowing us to discount stochastic cashflows at the risk-free rate. Unless otherwise stated all derivations involving stochastic processes are performed under $\mathbb{Q}$.

The EBIT process $\{Y_t\}$ includes downward discrete jump shocks and associated jump risk premia. We capture both jumps and jump risk premia by assuming that under $\mathbb{Q}$:

$$\frac{dY_t}{Y_{t-}} = \mu dt + \sigma d\mathcal{B}_t - (1 - Z)d\mathcal{J}_t, \quad Y_0 > 0, \quad (1)$$

where μ is the drift parameter, σ is the diffusion-volatility parameter, $\mathcal{B}$ is a standard Brownian motion process, and $\mathcal{J}$ is a pure jump process with a constant arrival rate λ under the risk-neutral measure.⁶ The inclusion of downward jumps generates negatively skewed and fat-tailed EBIT growth.

If a jump arrives at time t ($d\mathcal{J}_t = 1$), the firm's EBIT decreases from Y_{t-} to $Y_t = ZY_{t-}$, where $Z \in [0, 1]$ is a random variable drawn from a distribution function $F(\cdot)$. As long as the firm remains solvent, there is no limit to the number of these shocks; the occurrence of a jump shock does not change the likelihood of another in the future. The expected percentage EBIT loss upon a jump arrival is $(1 - \mathbb{E}(Z))$, where $\mathbb{E}[\cdot]$ is the expectation operator and the expected EBIT growth rate, g , is given by:

$$g = \mu - \lambda(1 - \mathbb{E}(Z)) . \quad (2)$$

The firm's financing sources involve both internal and external funds. Internal funds come from internally generated cash flows that are retained over time. External funds include short-term

⁶ In Appendix D, we derive the EBIT process (1) under the risk-neutral measure by starting with a jump-diffusion EBIT process under the physical measure and specifying risk premia for jump and diffusion shocks. The jump-diffusion process (1) generalizes the geometric Brownian motion process commonly used in the contingent-claim capital structure literature, e.g., Fischer, Heinkel, and Zechner (1989), Leland (1994a,b), and Goldstein, Ju, and Leland (2001). Malenko and Tsoy (2025) use this jump-diffusion EBIT process to study optimal time-consistent debt policy.

debt, which we assume to be costless for simplicity,⁷ and equity. Critically, external equity is costly in our model (and in reality). We denote by X_t the firm's debt balance at time t .

Let $C_t dt$ denote the (promised) debt interest payment over a small time interval $(t, t + dt)$. Conditional on not defaulting, the firm has to pay its outstanding debt balance (principal) X_t and interest $C_t dt$ at time $t + dt$.

Since debt matures instantaneously, both principal and interest must be repaid at every instant absent default. This is why short-term debt is always issued at par with an equilibrium credit spread. Following a large EBIT loss, shareholders may declare default. Under competitive debt markets, creditors price this default risk *ex ante*.

Equilibrium debt pricing. Let η_{t-} denote the equilibrium credit spread. Without loss of generality, we can write the interest payment C_{t-} as:

$$C_{t-} = (r + \eta_{t-})X_{t-}. \quad (3)$$

Creditors receive the firm's entire liquidation proceeds following default, but under limited liability creditors do not fully recover what is owed to them. To compensate debtholders for the credit risk they bear, the following expression for η_{t-} has to hold:

$$\eta_{t-} = \lambda \mathbb{E}_{t-} \left[\underbrace{\left(1 - \frac{L_t}{X_{t-}} \right)}_{\text{fractional LGD}} \mathbf{1}_t^{\mathcal{D}} \right], \quad (4)$$

where $\mathbf{1}_t^{\mathcal{D}} = 1$ if the firm defaults at t and $\mathbf{1}_t^{\mathcal{D}} = 0$ otherwise. Note that the credit spread η_{t-} depends on the firm's default policy. This is the channel through which the firm's endogenous default risk is priced by competitive, price-taking creditors. We discuss the economics of the credit spread in the next section. We provide a derivation of the credit risk pricing equation (4) in Appendix A.

Corporate taxes and earnings. In practice, corporate profits and losses have asymmetric tax treatments. We capture this feature in our model by assuming that the firm pays income taxes at

⁷ While in practice firms incur transaction costs in issuing debt or securing a line of credit, these costs are small relative to the costs firms incur when they issue equity. What is important for our analysis is that equity issuance costs are higher than debt issuance costs, not whether debt issuance is costly or not. This is why we set debt issuance costs to zero for simplicity.

a constant rate ($\tau > 0$) when it makes a profit ($Y_{t-} > C_{t-}$), and pays no taxes when it incurs a loss ($Y_{t-} \leq C_{t-}$). That is, the firm's tax payment Θ_{t-} is given by $\Theta_{t-} = \Theta(C_{t-}, Y_{t-}) = \theta(c_{t-})Y_{t-}$, where $c_{t-} = C_{t-}/Y_{t-}$ and

$$\theta(c) = \tau(1 - c)\mathbf{1}_{c \leq 1}, \quad (5)$$

is the scaled tax payment. The indicator function $\mathbf{1}_{c \leq 1}$ equals one if the firm makes a profit ($c \leq 1$) and zero otherwise (i.e., $c > 1$). We then obtain earnings, E_{t-} , by deducting interest payments $C_{t-} = (r + \eta_{t-})X_{t-}$ and taxes Θ_{t-} from EBIT Y_{t-} :

$$E_{t-} = \underbrace{Y_{t-}}_{\text{EBIT}} - \underbrace{(r + \eta_{t-})X_{t-}}_{\text{interest payments}} - \underbrace{\Theta_{t-}}_{\text{taxes}}. \quad (6)$$

As X_{t-} and Y_{t-} determine taxes and interest payments, they pin down earnings E_{t-} .

Financial distress costs. Following Leland (1994a) and the tradeoff theory literature, we assume that bankruptcy causes deadweight losses and specify the firm's recovery value at $T^{\mathcal{D}}$ as follows:

$$L_{T^{\mathcal{D}}} = \ell Y_{T^{\mathcal{D}}}, \quad (7)$$

where the multiple ℓ gives the firm's liquidation recovery per unit of EBIT.⁸ Unless ℓ is very high, the firm prefers operating over defaulting. Since the absolute priority rule (APR) applies in bankruptcy and since liquidation is inefficient, equityholders default only when it is no longer in their interest to continue operating the firm—that is, when the firm's equity value reaches zero.

Costly external equity issuance. Firms rarely issue external equity, and when they do, they incur significant costs, due to a combination of asymmetric information (Myers and Majluf, 1984), managerial incentive issues, and underwriting costs. For convenience we capture adverse selection costs, underwriting fees, and other flotation costs with a simple reduced form function. A large empirical literature has sought to quantify these costs, including the costs arising from the negative stock price reaction in response to the announcement of a new equity issue.⁹

⁸ We can gauge the magnitude of ℓ as in Leland (1994a). Suppose that the firm's liquidation value $L_{T^{\mathcal{D}}}$ equals a fraction $\varphi \in (0, 1)$ of its first-best value: $\Pi(Y) = \frac{Y}{r-g}$, where g is the growth rate given in (2). To ensure that $\Pi(Y)$ is finite, we require that $r > g$. Using $L_{T^{\mathcal{D}}} = \varphi \Pi(Y_{T^{\mathcal{D}}})$, we obtain $\ell = \varphi \Pi(Y)/Y = \varphi/(r - g)$.

⁹ Altinkilic and Hansen (2000) estimate underwriting fee schedules. Asquith and Mullins (1986) measure the indirect costs of external equity using seasoned equity offerings announcements. Hennessy and Whited (2007) use

Building on the findings of this literature, we assume that the firm incurs both fixed and proportional costs when it issues equity. Fixed costs are what generate the observed infrequent and lumpy equity issuance. As we will show, the proportional cost gives rise to two distinct targets for leverage: “optimal target leverage” when debt and earnings retention are the marginal sources of financing and the “recapitalization target” when external equity is the marginal source of financing. Only when fixed equity issuance costs are present do firms have an incentive to seek risk, causing firm value to become convex in the high-leverage region, as we demonstrate later.¹⁰

We assume that the fixed cost is proportional to Y_t , so that $h_0 Y_t$ denotes the total fixed equity-issuance cost. This formulation ensures that the firm never grows out of the fixed cost, preserving the first-order consideration of equity issuance costs and the model’s homogeneity property. The firm also incurs variable issuance costs: For each dollar the firm raises it incurs a marginal cost h_1 . Therefore, when raising net proceeds $M_t \geq 0$, the firm incurs a total equity issuance cost of $h_0 Y_t + h_1 M_t$. Let $dH_t = h_0 Y_t + h_1 M_t \geq 0$ denote this cost so that $\{H_t; t \geq 0\}$ is the cumulative (undiscounted) nondecreasing equity issuance cost process.

Equity payouts. In addition to managing debt and equity issuance, the firm also decides if and when to pay dividends. We denote by U_t the (undiscounted) cumulative payout to equityholders up to time t . In our model the optimal target leverage is closely tied to the firm’s payout decision to shareholders. Next, we introduce an accounting identity that is key to understanding the firm’s dynamic financial management.

Sources of funds equal uses of funds. Conditional on no default, the firm’s sources of funds must at all times equal its uses:

$$\underbrace{Y_t dt}_{\text{EBIT}} + \underbrace{(X_{t-} + dX_t)}_{\text{(gross) debt issuance}} + \underbrace{dN_t}_{\text{external equity}} = \underbrace{\Theta_t dt}_{\text{taxes}} + \underbrace{(X_{t-} + C_t dt)}_{\text{debt repayment}} + \underbrace{dU_t}_{\text{payouts}}, \quad (8)$$

where $dN_t \geq 0$ and $dU_t \geq 0$ denote the firm’s net external equity issuance and its incremental payout over $(t, t + dt)$, respectively. Note that dN_t and dU_t are two separate non-negative control

simulated method of moments to infer the magnitude of financing costs and their estimates support the view that external equity issuances involve large indirect costs and firms are sensitive to these costs.

¹⁰ In Appendix H, we analyze different effects of fixed versus proportional equity issuance costs, using a few helpful special cases of our model.

margins, with N_t and U_t denoting respectively the cumulative (undiscounted) non-decreasing external equity issuance and payouts up to time t . The second term on the left side of (8) represents the firm's gross debt issuance, which can be written as the sum of its balance due X_{t-} and incremental borrowing dX_t with no loss of generality, as dX_t can be arbitrarily chosen (positive or negative, smooth or lumpy). The three terms on the left side of (8) describe the sources of funds.

The three terms on the right side of (8) describe the uses of funds. In addition to paying taxes (the first term) and repaying its debt (the second term), the firm also decides whether to make a payout to shareholders or not (the third term).

Debt balances X_{t-} appear on both sides of (8). By eliminating X_{t-} on the two sides and rearranging terms, we obtain the following accounting identity for a solvent firm:

$$\underbrace{dU_t}_{\text{payouts}} = \underbrace{E_{t-} dt}_{\text{earnings}} + \underbrace{dX_t}_{\text{net debt issuance}} + \underbrace{dN_t}_{\text{external equity}}, \quad (9)$$

where E_{t-} represents earnings, as defined in (6). Equation (9) states that at all t , the cash outflows of a solvent firm (payouts) must equal the cash inflows, which arise from a combination of earnings, net debt issuance, and equity issuance. Importantly, out of the three policies: net debt issuance (dX_t), net equity issuance (dN_t), and payouts (dU_t), the firm can freely choose only two. The third must adjust as the residual to satisfy (9).

Since external equity is costly and a dollar dividend payment to shareholders is only worth a dollar, issuing equity to pay dividends is suboptimal. Therefore, a solvent firm can only be in one of the following three mutually exclusive scenarios: (a) issuing equity ($dN_t > 0$) but paying no dividends ($dU_t = 0$); (b) issuing no equity ($dN_t = 0$) but paying dividends ($dU_t > 0$); (c) neither issuing equity nor paying dividends ($dN_t = dU_t = 0$). In all three cases, the net debt issuance dX_t is uniquely pinned down by (9).¹¹

Next, we state the firm's optimization problem.

¹¹ Note that the credit spread η_{t-} internalizes all the benefits and costs of debt issuance for shareholders. Also there are no existing debtholders for shareholders to dilute because all outstanding debt has to be repaid before new financing can be obtained. Therefore, maximizing the firm's total value yields the same outcome as maximizing its equity value.

Firm optimality. Let $S = (T^D, \{N_t, U_t; t \in (0, T^D)\})$ denote the firm's (default timing, equity issuance, and equity payout) policies. The firm chooses its policy S to maximize its equity value:¹²

$$P(Y_t, X_t) = \mathbb{E}_t \left[\int_t^{T^D} e^{-r(s-t)} (dU_s - dN_s - dH_s) \right], \quad (10)$$

subject to the accounting identity (9) and the credit risk pricing equations (3)-(4). Note that we subtract equity issuance costs as $dH_t > 0$ whenever the firm issues equity, $dN_t > 0$. Shareholders only default when they receive zero at time T^D . We characterize the firm's optimal, time-consistent financing and default policies, taking as given creditors' competitive, zero-profit pricing of the firm's debt. See Appendices A and B for technical details.

Let $V_t = V(Y_t, X_t)$ denote firm value. As debt matures instantaneously, the market value of debt equals the face value of debt, so that:

$$V(Y_t, X_t) = P(Y_t, X_t) + X_t. \quad (11)$$

Let x_t denote the debt-EBIT ratio: $x_t = X_t/Y_t$, which is the natural state variable in our model.¹³ Let p_t denote the equity value-EBIT ratio ($p_t = P_t/Y_t$), and v_t denote the firm value-EBIT ratio ($v_t = V_t/Y_t = p_t + x_t$). Also let ML_t denote market leverage, the ratio of debt value X_t to firm value V_t : $ML_t = X_t/V_t = x_t/v_t$. In our model, $ML_t \in [0, 1]$. When a firm defaults, its market leverage equals one.

3 Dynamic Tradeoff Theory under Costless Equity

We first analyze the benchmark with costless external equity, $h_0 = h_1 = 0$. This case provides a dynamic version of the classic tradeoff theory that is fully dynamic and is closely related to Abel (2018). Our costless-external-equity benchmark also sets the stage for understanding how costly external equity fundamentally reshapes capital structure and leverage dynamics.

When external equity is costless, internal and external equity are perfect substitutes, so that there is no value in retaining earnings for future financing needs. Additionally, with

¹² We can extend our model to allow the firm's shareholders to be more impatient than creditors (see DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin, and Rochet (2007), DeMarzo and Fishman (2007), DeMarzo (2019), and Malenko (2019) among others, for models with this assumption.) See the NBER version of our paper for a model with this assumption.

¹³ This is consistent with Graham (2022), who reports that corporate capital structure appears to "focus on Debt/EBITDA or credit ratings rather than the classic debt/value or debt/assets."

(instantaneously maturing) short-term debt financing, the firm can switch between debt and equity at no cost, as long as it does not default. Therefore, the firm's scaled value, $v(x) = p(x) + x$, is time invariant in the solvent region and the optimal target debt-EBIT ratio, x^* , is constant. We can jointly pin down the optimal target debt-EBIT ratio, x^* , and equilibrium debt capacity, $\bar{x}$, above which the firm defaults. The analytical expressions for firm value $v(x)$, the optimal target x^* , and the equilibrium debt capacity $\bar{x}$ are derived in Internet Appendix IB.

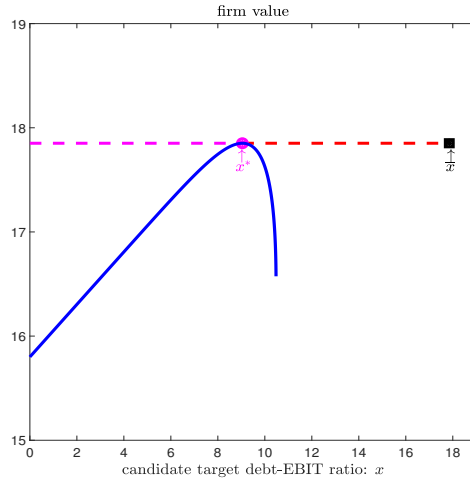


Figure 3: Dynamic tradeoff theory under costless equity issuance: $h_0 = h_1 = 0$. The optimal target debt-EBIT ratio x^* can be determined by plotting the scaled firm value for a range of candidate debt-EBIT ratio target x ; see the solid line.

We graphically illustrate the solution below. For this numerical solution, we specify a power function for the recovery fraction Z by setting $F(z) = z^\beta$ with $\beta = 9$. The other parameter values are set as follows: $r = 6\%$, $g = 1\%$, $\sigma = 40.6\%$, $\lambda = 1.2$, $\ell = 5.35$, and $\tau = 21\%$. See Appendix D for a detailed discussion of the calibration and parameter choices.

In Figure 3, we plot scaled firm value for a range of candidate debt-EBIT ratio targets x . Note that, since Y_t is stochastic and our model has the homogeneity property, the control variable is the debt-EBIT ratio rather than the level of debt used in the static tradeoff theory. The optimal target debt-EBIT ratio is $x^* = 9.05$, at which firm value is maximized: $v(x^*) = 17.9$ with an implied market leverage of $ML^* = 51\%$; see the magenta dot.¹⁴ Figure 3 exhibits a *dynamic*

¹⁴ From different, but complementary perspectives, the solid and horizontal dashed lines in this figure convey the same message: absent equity issuance costs, it is optimal for the firm to adjust x to the optimal target debt-EBIT

version of the classic (static) tradeoff theory solution (see, e.g., Figure 1 of Myers (1984), which pits the benefits of debt against the expected costs of default and Berk and DeMarzo (2020) for a textbook treatment).

Abel (2018) is the first to analyze a dynamic tradeoff model with *short-term debt* maturing instantaneously and with no external equity issuance costs. The key difference between Abel (2018) and the costless-equity special case of our model lies in the EBIT process. In Abel (2018), EBIT ϕ_t remains constant for an exogenous random duration and is then redrawn from a time-invariant distribution at Poisson arrival times. With this special EBIT process, Abel (2018) shows that the firm's optimal financing policy is characterized by an endogenous cutoff EBIT value, ϕ^* . When current EBIT ϕ_t is larger than the cutoff, $\phi_t > \phi^*$, the firm issues the same dollar amount of debt, D^* , independent of its EBIT.¹⁵ The implied time-series implication in Abel (2018) is that the firm never voluntarily reduces its outstanding debt and defaults when EBIT first falls below the cutoff ϕ^* if $\phi_0 > \phi^*$.

Our costless-equity benchmark yields a different debt adjustment policy because EBIT is a geometric process featuring downward jump arrivals. As a result, our model is homogeneous in EBIT and debt balance. The optimal policy can thus be expressed as a fixed target debt-EBIT ratio, x^* , rather than as a fixed dollar amount of debt as in Abel (2018). The firm adjusts the level of debt proportionally with EBIT in order to maintain a constant debt-EBIT ratio at x^* : following favorable EBIT shocks, the firm increases debt and makes payouts; following adverse but nondefaulting shocks, it issues equity to reduce debt and restore x_t to the target x^* . Hence, unlike in Abel (2018), the firm may voluntarily reduce its debt balance, and market leverage remains constant at ML^* .

In sum, despite featuring the same short-term debt as Abel (2018), the debt adjustment policies in our model are very different. Abel's EBIT process implies a fixed debt level for a solvent firm, except when market leverage is 100%, whereas our multiplicative EBIT process implies a constant target debt-EBIT ratio. In the next section, we reintroduce costly external

ratio x^* , for all $x \leq \bar{x} = 17.9$, implying that $v(x) = \bar{x}$ for all $x \leq \bar{x}$. When $x > \bar{x}$, the firm defaults, resulting in $v(x) = \ell = 5.35$. In effect, the distinction between state and control variables disappears when $h_0 = h_1 = 0$.

¹⁵ If current EBIT ϕ_t is below the endogenous cutoff, $\phi_t < \phi^*$, the firm uses 100% market leverage, resulting in zero equity value due to limited liability, until EBIT exceeds the cutoff ϕ^* .

equity, which generates persistent leverage and a wide target zone with state-contingent payout, recapitalization, and default policies.

4 Solution for the Baseline Model with Costly External Equity

In this section, we report the solution for our baseline model with costly external equity and discuss the mechanism. The firm dynamically optimizes all decision margins including issuance timing and size decisions for debt and equity, equity payouts, and default. At any time t , depending on its debt-EBIT ratio x_t the firm is in one of four possible regions, which we describe next.¹⁶

4.1 Optimal Target $\underline{x}$ and Payout Region ($dU_t > 0$ and $dN_t = 0$): $x < \underline{x}$

A firm with low or no debt (say at $t = 0$) wants to issue some debt because debt can shield taxes for shareholders. How much debt does the firm issue? Because of the model's homogeneity property, it is optimal for the firm to set x to $\underline{x}$, a time-invariant 'optimal target' for x . Since the firm with $x < \underline{x}$ can always increase its debt to $\underline{x}$ at no cost, the following value-matching condition holds:

$$p(x) = p(\underline{x}) + \underline{x} - x, \text{ for } x < \underline{x}. \quad (12)$$

Taking the limit $x \rightarrow \underline{x}$ in (12), we obtain the following smooth-pasting condition for $\underline{x}$:

$$p'(\underline{x}) = -1. \quad (13)$$

Since the target $\underline{x}$ is optimally chosen by shareholders, we also have the following super-contact condition (Dumas, 1991):

$$p''(\underline{x}) = 0. \quad (14)$$

Note that (14), implied by shareholders' equity value maximization, is also the counterpart to the first-order condition that pins down the optimal target leverage in static tradeoff theory (Myers, 1984): Firm value $v(x)$ is highest when x is at its optimal target $\underline{x}$. Unlike in the static tradeoff theory, however, it is not optimal for a firm to keep x at its optimal target $\underline{x}$ at all times. This is because x_t is exposed to (diffusion and jump) earnings shocks and issuing external equity

¹⁶ While the solution for the general case features four regions, the solutions for some special cases may feature fewer than four regions. Appendices G and H provide solutions of these special cases.

to keep x at $\underline{x}$ is too costly. In other words, to always instantaneously cover a loss with a new equity issue is far too costly.

4.2 Optimal Target Zone: Retained Earnings and Debt Financing ($dU_t = dN_t = 0$):

$$x \in [\underline{x}, \hat{x})$$

When x_t exceeds $\underline{x}$ but is still moderate, it is suboptimal for the firm to decrease x and bring it back to the ‘optimal target’ $\underline{x}$. This is because the only way to keep x at the target $\underline{x}$ is to issue external equity, which is costly. Instead, the firm optimally defers its equity issuance until x reaches $\hat{x}$ where the benefit of issuing external equity (weakly) outweighs the cost of doing so. As the debt-EBIT ratio x is already above the optimal target $\underline{x}$, the firm certainly does not want to voluntarily further increase its x by issuing debt to pay dividends. Therefore, $dN_t = dU_t = 0$ in this region.

The accounting identity (9) then implies that in the target zone, the increase of net debt issuance dX_t must exactly track the decrease of earnings:

$$dX_t = -E_{t-}dt = [(r + \eta_{t-})X_{t-} - Y_{t-} + \Theta_{t-}]dt. \quad (15)$$

That is, the optimal policy is to simply let earnings dictate debt dynamics. When making a profit (loss), that is, when earnings E_{t-} given in (6) are positive (negative), the firm’s debt balance automatically decreases (increases) by the amount $|E_{t-}|dt$. This is the zone where the firm spends most of its time. In effect, the firm adjusts its debt balance X_t upward (downward) in response to realized earnings losses (profits). This strategy can also be implemented through an exclusive long-term lending contract that specifies a state-contingent debt and interest-rate schedule *ex ante*. See subsection 4.6 for details.

Next, we describe leverage dynamics in the target zone. Applying Ito’s Lemma to $x_t = X_t/Y_t$ where X_t is given in (15) and Y_t is given in (1), we obtain:

$$dx_t = \mu_x(x_{t-})dt - \sigma x_{t-}d\mathcal{B}_t + (x_t^{\mathcal{J}} - x_{t-})d\mathcal{J}_t, \quad (16)$$

where $\mu_x(\cdot)$ is the drift of x given by:

$$\mu_x(x_{t-}) = [c(x_{t-}) + \theta(c(x_{t-})) - 1] - (\mu - \sigma^2)x_{t-}. \quad (17)$$

The interest payment per unit of EBIT, $c(x_{t-})$, is given by:

$$c(x_{t-}) = (r + \eta(x_{t-}))x_{t-} , \quad (18)$$

where $\eta(\cdot)$ is the equilibrium credit spread given by:

$$\eta(x_{t-}) = \lambda \underbrace{\int_0^{x_{t-}/\bar{x}} \left(1 - \frac{\ell z}{x_{t-}}\right) dF(z)}_{\text{fractional LGD}} . \quad (19)$$

Note that we obtain (19) by using the homogeneity property to simplify (4).

The intuition for (19) is as follows. First, creditors' stochastic fractional loss given default (LGD) equals $1 - (L_t/X_{t-}) = 1 - \ell Z/x_{t-}$. Second, note that a jump arrival is necessary but not sufficient to trigger default. Only when creditors' LGD is large enough ($Z < x_{t-}/\bar{x}$) will the firm default. These observations together explain (4) and (19). As a special case, if default yields no recovery ($\ell = 0$), the credit spread is: $\eta(x_{t-}) = \lambda F(x_{t-}/\bar{x})$.

The $-\sigma x_{t-}$ diffusion coefficient in (16) captures the effect of EBIT diffusion shocks on x_{t-} : A positive shock increases the firm's value and lowers x explaining the negative sign.¹⁷ The last term in (16) describes the effect of a jump shock on x . Upon arrival of a jump loss at t , the debt-EBIT ratio changes from x_{t-} to $x_t^J = x_{t-}/Z$. To account for the effect of jumps on x , we introduce the following:

$$g_x(x_{t-}) = \mu_x(x_{t-}) + \lambda \left[\mathbb{E}_{t-} \left(\frac{x_{t-}}{Z} \right) - x_{t-} \right] , \quad (20)$$

where $\mu_x(x_{t-})$ is given in (17).

Next, we characterize the equity value $p(x)$. In Appendix B.1, we show that for $x \in (\underline{x}, \hat{x})$, $p(x)$ satisfies the following integro-differential equation:

$$(r - \mu) p(x) = [c(x) + \theta(c(x)) - 1 - \mu x] p'(x) + \frac{\sigma^2 x^2}{2} p''(x) + \lambda \mathbb{E} [Z p(x/Z) - p(x)] . \quad (21)$$

The first two terms on the right side of (21) capture the drift and volatility effects of x on $p(x)$. The last term in (21) captures the effect of jumps on equity value. A jump causes the

¹⁷ Within the interior of this region, the debt balance satisfies $dX_t = -E_{t-}dt$, whereas the debt-EBIT ratio inherits Brownian fluctuations from EBIT. The distinction from DeMarzo and He (2021) therefore concerns adjustment incentives rather than the local order of debt changes: our firm uses retained earnings to reduce debt and undertakes costly equity-financed recapitalization at high leverage, whereas their shareholders never voluntarily repurchase outstanding debt.

debt-EBIT ratio to increase from x_{t-} to $x_t^J = x_{t-}/Z$ and the equity value to decrease from $p(x_{t-})$ to $p(x_t^J) = p(x_{t-}/Z)$.

Although the firm relies on earnings retention and debt rollover throughout the target zone, its financial situation is very different depending on the level of x . Near the left payout boundary $\underline{x}$, x_t drifts downward in expectation and the firm has a strong balance sheet. In contrast, near the right boundary $\widehat{x}$, x_t drifts upward at a fast rate and the firm has a vulnerable balance sheet. Importantly, earnings growth jumps can not only cause default but also generate rich leverage dynamics.

Absent jumps, our model only features continuous, diffusive shocks and can be solved using standard finite-difference methods requiring only local information around x . In contrast, in the presence of jump shocks a global solution method is required because $p(x)$ is linked to $p(x/Z)$ for all possible values of the random recovery $Z \in [0, 1]$. In short, the economics of the jump-diffusion model are much richer, but solving this model requires more involved numerical solution methods.

Next, we turn to the region where the firm proactively issues equity to deleverage.

4.3 Recapitalization and Deleveraging Region ($dN_t > 0$ and $dU_t = 0$): $x \in [\widehat{x}, \bar{x}]$

When x rises above an endogenous boundary $\widehat{x}$, the firm recapitalizes its balance sheet to deleverage ($dN_t > 0$) even though this involves a costly seasoned equity offering. As long as the cost of issuing external equity is low or moderate, a recapitalization is valuable as it reduces the expected cost of an even worse outcome in the future: a costly default.

Let $m_t = M_t/Y_t \geq 0$ denote the (scaled) net equity issuance proceeds. Let $\widetilde{x}_t = \widetilde{X}_t/Y_t$ denote the ‘*recapitalization target*’, that is, the debt-EBIT ratio immediately after the firm has delevered through the recapitalization:

$$\widetilde{x}_t = x_t - m_t. \quad (22)$$

Given that $p(x)$ must be continuous before and after issuance, the following value-matching condition holds:

$$p(x) = p(\widetilde{x}) - [h_0 + (1 + h_1)(x - \widetilde{x})] \quad \text{for } x \in [\widehat{x}, \bar{x}]. \quad (23)$$

Equation (23) states that the firm’s pre-issuance equity value $p(x)$ must equal its post-issuance equity value, $p(\widetilde{x})$, after accounting for: (a.) the fixed equity issuance cost h_0 ; (b.) the variable

issuance cost $h_1 m_t$; and (c.) the net equity issuance amount m_t . Differentiating (23) with respect to x yields

$$-p'(x) = 1 + h_1 \quad \text{for } x \in [\hat{x}, \bar{x}], \quad (24)$$

which equates the marginal benefit with the marginal cost of issuing equity.

Also, since the firm optimally reaches the recapitalization target $\tilde{x}$ after issuing equity, $-p'(\tilde{x}) = 1 + h_1$ must hold. As (23) and (24) hold at the firm's proactive deleveraging threshold $\hat{x}$, we summarize the boundary conditions for $\hat{x}$ and $\tilde{x}$ as follows:

$$p(\hat{x}) = p(\tilde{x}) - [h_0 + (1 + h_1)(\hat{x} - \tilde{x})] \quad \text{and} \quad -p'(\hat{x}) = -p'(\tilde{x}) = 1 + h_1. \quad (25)$$

Importantly, the 'equity recapitalization target' $\tilde{x}$ is different from the 'optimal leverage target' $\underline{x}$ where firm value $v(x)$ is maximized. The firm deleverages its balance sheet from the $x \geq \hat{x}$ region to $\tilde{x}$ in the target zone by issuing costly equity. In contrast, the firm reaches the 'optimal leverage target', $\underline{x}$, when its marginal cost of issuing debt is zero: $v'(\underline{x}) = 0$ so that at the margin the firm is indifferent between paying a dividend to shareholders or not when it chooses to move to $\underline{x}$. In sum, the 'equity recapitalization target' $\tilde{x}$ and the 'optimal leverage target' $\underline{x}$ are two different targets and $\underline{x} < \tilde{x}$ since the marginal cost of debt issuance is lower than the marginal cost of equity issuance ($h_1 > 0$).

Our model highlights that the idea of optimal leverage in a dynamic setting cannot be reduced to the concept of a single target ratio, as the existing literature has done. First of all, the correct concept is a target zone in which leverage evolves through changes in debt in response to profits and losses. And second, when leverage exits this region, there are two distinct return points, one on the left (the optimal leverage target) when leverage exits the target zone on the left, and one on the right (the optimal recapitalization target) when leverage exits the target zone on the right.

Being able to issue costly external equity to proactively deleverage its balance sheet is a valuable option for a firm managing its leverage dynamics. Overall, the equity issuance option generates highly nonlinear and asymmetric leverage dynamics even when the firm is operating in the target zone.

4.4 Equilibrium Debt Limit $\bar{x}$ and Default Region: $x > \bar{x}$

The firm only uses default as a last resort. When the firm's debt-EBIT ratio x is so large that it exceeds an endogenously determined level $\bar{x}$, the firm defaults. Because of limited liability and APR, equity value equals zero:

$$p(x) = 0 \quad \text{for } x \geq \bar{x}, \quad (26)$$

and creditors collect the firm's liquidation value ℓ per unit of EBIT. As $\bar{x}$ is the lowest value of x above which the firm is unable to refinance its debt and to continue as a going concern, we refer to it as the equilibrium debt limit (or capacity).

Since the firm is indifferent between defaulting and issuing equity when $x = \bar{x}$, we can pin down the equilibrium debt limit $\bar{x}$ and the equity recapitalization target $\tilde{x}$ using the following two equations:

$$p(\tilde{x}) = h_0 + (1 + h_1)(\bar{x} - \tilde{x}) \quad \text{and} \quad p(\bar{x}) = 0. \quad (27)$$

The intuition for the first equation in (27) is as follows. The right side is the sum of (a.) the maximal (net) amount that the firm can raise via external equity: $m = \bar{x} - \tilde{x}$ and (b.) the associated equity issuance costs $(h_0 + h_1 m)$. Immediately after the firm issues equity when $x = \bar{x}$, old shareholders are completely wiped out so that new shareholders own the entire firm's equity, valued at $p(\tilde{x})$. This explains the first equation in (27). Together with the zero-equity value condition at $x = \bar{x}$, we can jointly pin down the debt limit $\bar{x}$ and the equity recapitalization target $\tilde{x}$.

4.5 Solution Summary and Graphical Illustration

In summary, the solution for the general case features four regions, divided by three endogenous thresholds, $\underline{x} < \hat{x} \leq \bar{x}$. The firm makes payouts when $x < \underline{x}$, retains earnings and relies on debt as the marginal source of financing when $x \in [\underline{x}, \hat{x})$, issues equity when $x \in [\hat{x}, \bar{x}]$, and defaults when $x > \bar{x}$. We provide a formal statement of this solution (and the conditions under which it holds) in Theorem 4 in Appendix B. We provide a proof of Theorem 4 in the Internet Appendix IA.

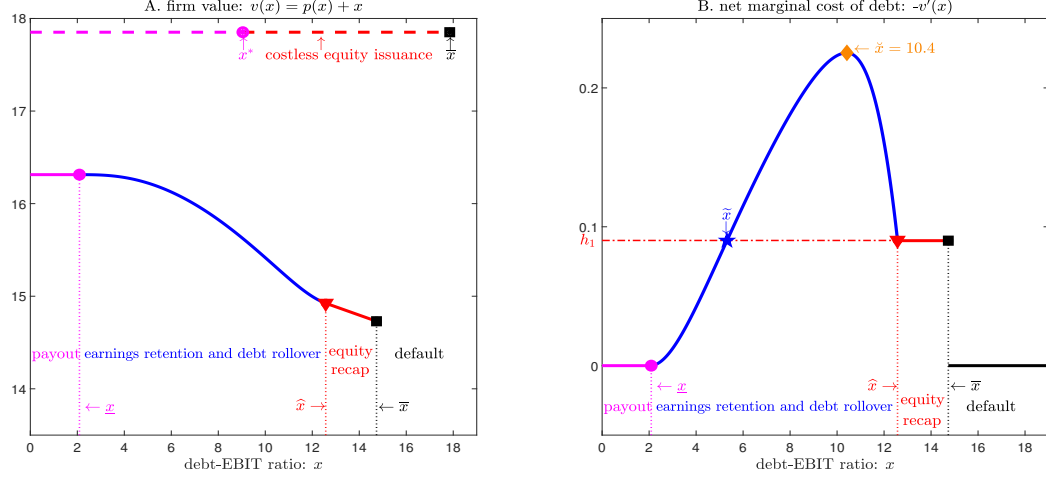


Figure 4: Firm value $v(x)$ and net marginal cost of debt $-v'(x)$ under costly external equity. The target debt-EBIT ratio is $\underline{x} = 2.1$, the proactive deleveraging threshold is $\hat{x} = 12.6$, and the equilibrium debt capacity is $\bar{x} = 14.7$.

A quantitative illustration of the solution is given below. The parameter values for this solution are as follows. For equity issuance costs, we set $h_0 = 0.62$ and $h_1 = 0.09$, which are in line with estimates in the literature (Hennessy and Whited, 2007; Eckbo et al., 2007; Bolton et al., 2011). For all other parameters, we use the same values as in Section 3.

Panel A of Figure 4 plots firm value $v(x) = p(x) + x$.¹⁸ It covers four regions separated by: i) the target debt-to-EBIT ratio $\underline{x}$ (the magenta dot); ii) the proactive deleveraging threshold $\hat{x}$ (the red triangle); and iii) the equilibrium debt capacity $\bar{x}$ (the black square). To the left of the magenta dot, firm value is constant: $v(x) = 16.3$ (the magenta horizontal line). This is because shareholders can increase leverage at no cost by moving x to $\underline{x}$, leaving firm value invariant with respect to x for all $x \leq \underline{x}$. Between the magenta dot and the red triangle, the firm is in the earnings retention and debt financing region in which $v(x)$ is nonlinear and decreasing in x , reflecting the rising expected future equity issuance costs as x increases. Between the red triangle and the black square, the firm proactively issues equity to deleverage its balance sheet. To the right of the black square, the firm is no longer solvent and defaults.

¹⁸ Reporting the solution in terms of $v(x)$ rather than $p(x)$ involves no loss of information because debt is continuously issued and priced at par, resulting in $v(x) = p(x) + x$. It is also more reader-friendly and facilitates comparison with the costless equity issuance case, in which (scaled) firm value is constant, provided that it is solvent.

Within the target zone, the firm issues no external equity and makes no payouts; instead, leverage evolves through earnings retention and debt rollover. Positive earnings shocks are used to reduce debt, producing a negative relation between profitability and leverage.¹⁹ The target zone is a wide interval $(\underline{x}, \widehat{x}) = (2.1, 12.6)$, within which leverage can vary substantially with the firm’s earnings history. Unlike target-leverage models, in which equity issuance and payouts are frequently used to return leverage to a particular target, our costly-external-equity model instead generates this wide target zone endogenously, consistent with the evidence. For example, DeAngelo and Roll (2015) state that “Leverage models that place little or no weight on maintaining a particular ratio do a good job replicating the substantial instability of the actual leverage.”²⁰

Once x_t enters the recapitalization region, $x \in [\widehat{x}, \bar{x}] = [12.6, 14.7]$, the firm proactively issues equity to deleverage its balance sheet. This deleveraging prediction is consistent with survey evidence suggesting that CFOs attach considerable importance to keeping leverage within limits in order to maintain their credit ratings (Graham and Harvey, 2001). Section 5.1 provides further discussion of the model’s predictions about deleveraging propensities.

For comparison, when external equity issuance is costless, both firm value and the debt capacity equal $\bar{x} = 17.9$, so that firm value is 17.9 for all $x \leq \bar{x} = 17.9$ (see Section 3). The gap between the dashed horizontal line and the solid line therefore measures the value loss from costly equity issuance. This gap widens with x as expected future issuance costs increase.

Panel B of Figure 4 further shows that the net marginal cost of debt, $-v'(x)$, is non-monotonic within the target zone. It increases over $x \in (\underline{x}, \check{x}) = (2.1, 10.4)$, reaching its maximum of 0.225 at the inflection point $\check{x} = 10.4$, where $v''(x) = 0$ and market leverage is $ML(\check{x}) = \check{x}/v(\check{x}) = 68\%$. Thus, $v(x)$ is concave and the firm is endogenously risk averse in this $(\underline{x}, \check{x})$ subregion. In the subregion where $x \in (\check{x}, \widehat{x}) = (10.4, 12.6)$, $-v'(x)$ decreases, implying that $v(x)$ is convex and the firm becomes endogenously risk seeking. For $x \in (\widehat{x}, \bar{x}) = (12.6, 14.7)$, $-v'(x) = h_1 = 0.09$, as the firm incurs a fixed equity-issuance cost (h_0) and a marginal equity-issuance cost (h_1) to raise $m = x - \widetilde{x}$ in net (scaled) proceeds.

¹⁹ See Titman and Wessels (1988) and Rajan and Zingales (1995) on the leverage-profitability puzzle. Denis and McKeon (2012) show that leverage dynamics depend on whether a firm generates a financial surplus, while DeAngelo, Gonçalves, and Stulz (2018) find that firms reduce debt following positive earnings shocks and increase debt when necessary.

²⁰ See, e.g., Graham and Harvey (2001), Fama and French (2002, 2005), Leary and Roberts (2005), and Lemmon, Roberts, and Zender (2008).

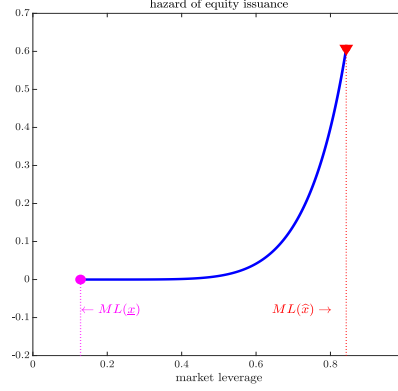


Figure 5: Hazard of equity issuance as a function of market leverage $ML(x)$.

A key feature of our model is the significant nonlinearity due to costly external equity financing. To illustrate the highly nonlinear equity decisions, we plot the hazard rate of equity issuance in Figure 5. In addition to being highly nonlinear, our model also predicts that equity issuance is uncommon, occurring with a frequency of 3.7% per year, which is close to the estimated likelihood of 3.4% per year for seasoned equity offering (SEO) reported in DeAngelo, DeAngelo, and Stulz (2010).²¹ A key takeaway is that even though a firm (unconditionally) rarely uses external equity financing, its likelihood of issuing external equity increases significantly with leverage once the firm becomes moderately levered, fundamentally altering its state-contingent interdependent policies.

Finally, Table 1 summarizes four commonly used measures of leverage: the target debt-EBIT ratio $\underline{x} = 2.1$, the recapitalization target $\tilde{x}$, the proactive deleveraging threshold $\hat{x} = 12.6$, and the equilibrium debt capacity $\bar{x} = 14.7$.

As long as market leverage (ML) lies between 13% and 84%, or equivalently, its interest coverage ratio (ICR) lies between 8.0 and 0.3, the firm is in the earnings retention and debt financing region, neither issuing equity nor paying dividends. If its ML falls below 13%, the firm pays dividends to raise its ML to 13% and lower its ICR to 8.0. If its ML suddenly exceeds 84% but remains below 100%—for example, after it is hit by a negative earnings jump—it issues external

²¹ The hazard rate of equity issuance under the physical measure $\mathbb{P}$ is given by $\lambda^{\mathbb{P}} \mathbb{P}_{t-} \left(\frac{x_{t-}}{\bar{x}} \in [\hat{x}, \bar{x}] \right) = \lambda^{\mathbb{P}} \left[F\left(\frac{x_{t-}}{\bar{x}}\right) - F\left(\frac{x_{t-}}{\hat{x}}\right) \right]$ for $x_{t-} \in (\underline{x}, \hat{x})$, where $\lambda^{\mathbb{P}} = 0.8$ is the jump arrival rate under $\mathbb{P}$. See Appendix D for the change of measure, calibration details, and the simulation-based computation of annual equity-issuance frequency.

Table 1: MARKET LEVERAGE AND INTEREST COVERAGE AT KEY THRESHOLDS. This table reports market leverage, $ML = x/v(x)$, and the interest coverage ratio, $ICR = 1/[(r + \eta(x))x]$, at four key debt-EBIT thresholds: the target debt-EBIT ratio $\underline{x} = 2.1$, the recapitalization target $\tilde{x} = 5.3$, the proactive deleveraging threshold $\hat{x} = 12.6$, and the equilibrium debt capacity $\bar{x} = 14.7$.

Debt-EBIT Ratio x	$\underline{x} = 2.1$	$\tilde{x} = 5.3$	$\hat{x} = 12.6$	$\bar{x} = 14.7$
$ML = x/v(x)$	13%	33%	84%	100%
$ICR = 1/[(r + \eta(x))x]$	8.0	3.1	0.3	0.1

equity to deleverage its balance sheet and reduce its ML to 33%. If its debt-EBIT ratio exceeds $\bar{x}$ (or equivalently, its ICR is below 0.1), the firm defaults.

4.6 Equivalent Long-Term Lending Contract Implementation

There is an alternative exclusive long-term lending contract formulation that yields the same outcome as the equilibrium solution for the baseline model of Section 2.

Long-term lending contract. At time 0, the firm enters a long-term exclusive contract with a single lender on the following terms. The lender funds the initial balance X_0 and commits to funding the firm's borrowing X_t as long as $X_t \leq \bar{x}Y_t$, where $\bar{x}$ is the debt capacity. The firm pays a state-contingent interest rate $(r + \eta(x))$ where $x \in [0, \bar{x}]$. Both the debt capacity, $\bar{x}$, and the credit-spread schedule, $\eta(x)$, are determined at $t = 0$.

Real-world syndicated credit agreements — such as revolving credit facilities and term loan A tranches — resemble our long-term lending contract formulation. Typically, this type of credit facility features a leverage-based pricing grid, under which the interest-rate margin over a reference rate (e.g., the secured overnight financing rate, SOFR) is fixed *ex ante* as a schedule of tiers indexed to a leverage ratio (typically total net debt to EBITDA), and the borrower's margin resets to the applicable tier upon delivery of periodic (typically quarterly) compliance certificates. Since the pricing grid is committed *ex ante*, rather than renegotiated loan by loan or left to the lender's discretion, it captures the borrower's future leverage-dependent credit risk in advance, just as the equilibrium credit spread $\eta(x)$ does in our long-term contract.²²

²² See, e.g., Asquith, Beatty, and Weber (2005) on the design of leverage- and rating-based pricing grids in bank debt contracts, and Ivashina and Kovner (2011) on loan terms, including covenant structure, in private-equity-sponsored leveraged buyout financing. There are two institutional features of these market pricing grids that differ from our long-term contract: the margin schedule is a step function over a small number of leverage tiers rather than the continuous schedule $\eta(x)$, and the tier reset occurs at discrete (e.g., quarterly) intervals rather than continuously.

We show in Appendix C that the equilibrium outcome of the short-term debt formulation can be implemented through this long-term contract.²³ Moreover, the equityholders' optimization problem at $t = 0$ under the long-term contract is time consistent. That is, even if equityholders could re-optimize at any time t , they have no incentive to choose policies different from those under commitment. By contrast, in DeMarzo and He (2021), shareholders cannot sustain the debt policy that would be optimal under commitment because they have incentives to issue additional debt and dilute outstanding creditors' claims. Their equilibrium debt policy is itself time consistent.

5 Empirical Predictions

We now undertake a quantitative analysis that evaluates detailed model predictions against empirical findings in the literature. Overall, we find that the calibrated model generates empirically plausible predictions about corporate deleveraging propensities, predicted market leverage, credit spreads, the equity-issuance frequency and default risk.²⁴

5.1 Corporate Deleveraging

DeAngelo, Gonçalves, and Stulz (2018) document that firms often deleverage from their historical peak market leverage (ML) to much more conservative levels. To assess whether our model reproduces this feature of corporate deleveraging, we simulate firm-level ML paths under the physical measure and construct peak-to-trough deleveraging episodes following DeAngelo, Gonçalves, and Stulz (2018). For each simulated path, we identify peak ML, the subsequent trough, the corresponding peak-to-trough change in ML, and the number of years between the peak and the trough. See Appendix D.4 for details of the simulation procedure.

Table 2 reports the medians of these four statistics across all simulated paths and compares them with the corresponding empirical estimates in DeAngelo, Gonçalves, and Stulz (2018). Our

²³ This implementation equivalence is related to Fudenberg, Holmstrom, and Milgrom (1990), who identify conditions under which a long-term agency arrangement can be implemented through a sequence of short-term contracts.

²⁴ Unless otherwise stated, for our analysis in this section, we use the same parameter values as in subsection 4.5: $r = 6\%$, $g = 1\%$, $\sigma = 40.6\%$, $\lambda = 1.2$, $\beta = 9$, $\ell = 5.35$, $\tau = 21\%$, $h_0 = 0.62$, and $h_1 = 0.09$. See Appendix D for a detailed discussion of the calibration. The equity-issuance frequency is discussed in subsection 4.5.

Table 2: DELEVERAGING EPISODES. The table reports medians of market leverage (ML) at peak, the subsequent trough, the peak-to-trough change in ML, and the peak-to-trough duration. Details on the calculation of the model-implied moments in the second column are provided in Appendix D.4. The empirical moments in the third column are from Tables 2 and 3 of DeAngelo, Gonçalves, and Stulz (2018).

	Model-implied moments	Empirical moments
Median ML at peak	53.4%	54.3%
Median ML at trough after peak	12.8%	2.6%
Median peak-to-trough decline in ML	−39.5%	−39.5%
Median years from ML peak to trough	7	6

calibrated model predicts (1) a median peak ML of 53.4%, compared with 54.3% in the data; (2) a median peak-to-trough decline of 39.5%, matching the data; and (3) a median peak-to-trough transition of seven years, compared with six years in the data. However, the calibrated model predicts a higher subsequent trough ML of 12.8%, compared with 2.6% in the data. Overall, the simulated deleveraging episodes are broadly consistent with the findings in DeAngelo, Gonçalves, and Stulz (2018).

5.2 Credit Spread and Debt Spiral

Figure 6 plots the equilibrium credit spread and the drift of the debt-EBIT ratio x . Panel A shows that the credit spread η is close to zero when ML is low or moderately low. For example, when ML=30%, the credit spread is only 0.004%. However, credit spreads are high and sensitive to changes in leverage when ML is moderate or high. For example, as ML rises from 50% to 60%, the credit spread increases more than fourfold, from 0.3% to 1.3%. Additionally, equilibrium credit spreads predicted by our calibrated model are broadly in line with the observed one-year CDS spreads. We show this result in Panel A of Figure 6 by plotting the one-year CDS spreads reported by Huang, Shi, and Zhou (2020), displayed as black dots, and comparing them with our model-implied credit spreads.²⁵

Why do credit spreads increase so quickly as leverage increases from moderate to moderately high levels? The reason is that interest payments can quickly increase as credit risk rises with leverage and a “debt-begets-debt” spiral kicks in. Moreover, an empirically plausible jump risk premium further increases the cost of financial distress (Almeida and Philippon, 2007). Taken

²⁵ We use one-year CDS spreads as a proxy for credit spreads on bonds with short maturities.

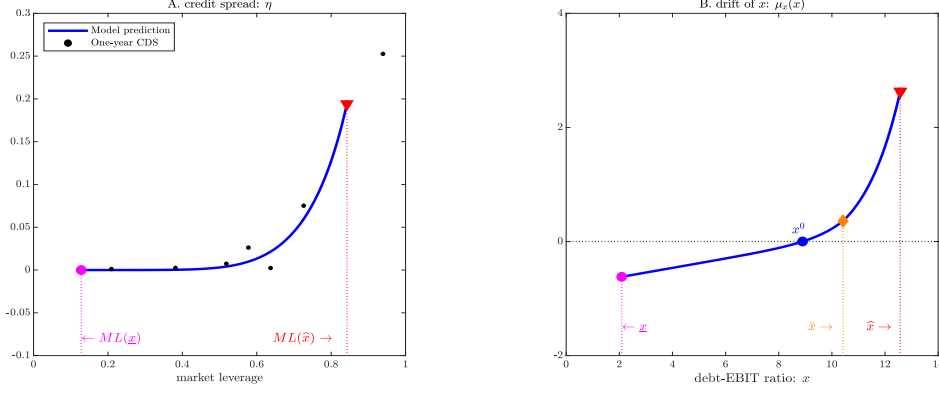


Figure 6: Equilibrium credit spread, η , and drift of x , $\mu_x(x)$, in the retained earnings and debt financing region: $x \in [\underline{x}, \hat{x}]$.

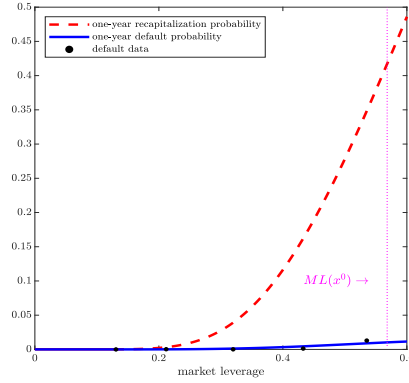


Figure 7: One-year conditional probabilities of entering the recapitalization region, $x \in [\hat{x}, \bar{x}]$, and the default region, $x > \bar{x}$.

together, large earnings shocks, the associated jump risk premium, and costly external equity induce a financially constrained firm to be prudent by seeking to maintain low leverage.

Panel B of Figure 6 plots the drift $\mu_x(x)$ of the debt-EBIT ratio between jumps, given in (17). Within the target zone, this drift changes sign at $x^0 = 8.9$: it is negative for $\underline{x} < x < x^0$ and positive for $x^0 < x < \hat{x}$. The latter interval is the debt-spiral region. Higher leverage raises the equilibrium credit spread, increasing interest expenses and borrowing needs; additional borrowing can, in turn, further raise leverage and credit spreads. This feedback contributes to the upward drift of the debt-EBIT ratio. The sign of $\mu_x(x)$ describes the ratio's local drift between jumps, rather than the change in the debt balance itself; Brownian shocks, jumps, and financing interventions also shape realized leverage paths. Because equity issuance is costly, the

firm continues to retain earnings and roll over debt within this region. When leverage enters the recapitalization region, $x \in [\hat{x}, \bar{x}]$, where $\hat{x} = 12.6$, the firm issues equity and reduces debt. In the calibrated simulation, the firm spends only about 3% of its time in the debt-spiral region. The financing-cost feedback nevertheless helps explain the state-dependent leverage dynamics and credit-risk predictions.

To further shed light on the highly nonlinear dynamics of leverage and credit risk, Figure 7 plots the one-year conditional probabilities of entering the recapitalization region, $x \in [\hat{x}, \bar{x}]$, and the default region, $x > \bar{x}$. The model-implied one-year default probabilities are broadly consistent with the empirical default rates reported by Huang and Huang (2012), which are shown as black dots in Figure 7. The figure also shows that the conditional probabilities of equity issuance are highly nonlinear, consistent with the nonlinear patterns in credit spreads η and leverage drift μ_x . Finally, the default probability is negligible relative to the probability of equity issuance. This is because firms often choose to issue equity, thereby significantly diluting existing shareholders, before declaring default. Thus, costly equity financing is an integral part of financial distress management and an important determinant of leverage dynamics.

In summary, our model generates highly nonlinear leverage and credit-risk dynamics, which relate to Brunnermeier and Sannikov (2014)’s key result that highly nonlinear amplification causes the economy to be prone to instability and to occasionally enter volatile crisis episodes.

5.3 Comparative Dynamics of Leverage

Our model also generates predictions for market leverage that are in line with other evidence. Using simulated paths, we can compute mean market leverage based on the pooled pre-default occupation distribution, which weights each pre-default firm-period observation equally across all simulated paths; see Appendix D.3 for details. The resulting mean market leverage is approximately 23%, consistent with an average market leverage of 20%-25% for US data for listed companies (Graham, 2000; Strebulaev and Whited, 2012)

We use comparative dynamics to understand the determinants of leverage. Two surveys of CFOs of large US listed companies conducted about twenty years apart, Graham and Harvey (2001) and Graham (2022) consistently find that CFOs list financial flexibility, the firm’s credit

Table 3: EFFECTS OF EQUITY ISSUANCE COSTS ON MARKET LEVERAGE.

	Mean	Std. dev.	5%	50%	95%
baseline	0.23	0.12	0.13	0.19	0.50
$h_0 = 0$	0.34	0.13	0.19	0.30	0.61
$h_0 = 2$	0.16	0.10	0.10	0.12	0.36
$h_1 = 0$	0.30	0.12	0.19	0.26	0.56
$h_1 = 0.3$	0.16	0.10	0.09	0.12	0.39

rating, and earnings risk as the top three concerns that influence their debt policy decisions. The predictions of our model line up well with these three factors. We can show this through several comparative dynamics of market leverage with respect to (i) equity issuance costs and (ii) jump risk parameterized by λ and β . We also show that the key parameters in the standard tradeoff theory, the tax rate τ and especially the liquidation recovery value parameter ℓ , have limited quantitative effects on leverage, which is consistent with the findings of the two surveys mentioned above.

5.3.1 Leverage Decreases with Equity Issuance Costs

A static theory would predict that when the cost of equity increases, firms rely more on debt financing. In contrast, Table 3 shows that as we increase equity issuance costs (h_0 or h_1), leverage *decreases*. As external equity becomes more costly to issue, the firm faces a higher cost of recapitalization down the road, which in turn discourages its *ex ante* debt issuance.

Table 3 also shows that the model’s low-leverage prediction remains robust across a wide range of equity-issuance cost parameter values. In Appendix F, we further show that the changes in market leverage reported in Table 3 are primarily driven by shifts in the density of the debt-EBIT ratio, rather than by changes in the mapping from the debt-EBIT ratio x to market leverage $ML(x)$. Note that the presence of equity issuance costs is key. When equity issuance costs are turned off, market leverage is constant and substantially higher, at 51% (see Figure 3). In sum, a key takeaway of our analysis is that equity issuance costs have first-order conceptual and quantitative effects on leverage.

5.3.2 Effects of Earnings Jump Risk

Overall, the effects of disaster jump risk on market leverage are also sizeable. The reason is that the firm has few options for adjusting its leverage in response to large downward jump shocks. As with the effect of equity issuance costs, on average, market leverage is low for a wide range of jump parameter values; see Appendix F for details.

5.3.3 Limited Tax and Financial Distress Cost Effects on Leverage

How much do changes in the corporate tax and bankruptcy costs affect corporate leverage in our model? Overall, we find that changes in the corporate tax rate have some, but very limited, effects on market leverage. These effects are much smaller than those predicted in the contingent-claims capital structure literature following Leland (1994a). For example, when the tax rate τ is increased from 21% to 30%, median market leverage increases from 19% to only 22%. These limited responses of corporate leverage policies to significant changes in corporate tax rates are in line with the evidence (Graham, 2000). Changes in the default recovery have an even smaller effect on leverage. Appendix F provides a more detailed analysis.

Revisiting Miller (1977)’s famous “rabbit” versus “horse” analogy, we see that neither tax rates nor distress costs are horses once we take external equity financing costs into account.

6 A Model with both Cash-holding and Debt Regions

In this section, we extend our baseline model to a setting where the firm may choose to hold precautionary *savings* under certain circumstances. The key departure from the baseline model is that operating cash flow is also subject to a contemporaneous shock. Specifically, whereas the firm’s risk-adjusted EBIT over $(t, t + dt)$ is $Y_t dt$ in the baseline model, it is now given by

$$Y_t (dt + \nu dZ_t), \tag{28}$$

where Z_t is a newly introduced standard Brownian motion and $\nu > 0$ measures the volatility of the contemporaneous cash-flow shock. Let ρ denote the correlation coefficient between Z_t and the Brownian motion $\mathcal{B}_t$ that drives the Y_t process.²⁶ In this extended model, $Y_t dt$

²⁶ Mathematically, we introduce a bivariate Brownian motion $(\mathcal{B}_t, Z_t)$ under the risk-neutral measure.

should be interpreted as *expected* EBIT. In this extension of the EBIT process, with additional contemporaneous cash-flow risk, the firm may have an incentive to hold cash.

Under this extension, debt balance evolves as:

$$dX_t = (C_{t-} + \Theta_{t-})dt - Y_{t-}(dt + \nu dZ_t) - dN_t + dU_t. \quad (29)$$

Given that the shock term νdZ_t directly enters the firm's contemporaneous cash flow, the firm's debt balance X_t is now a controlled diffusion process. This randomness can cause X_t to turn negative, so that the firm holds cash with a balance of $-X_t$. With these preliminaries in place, we can now analyze the firm's precautionary saving motive.

Consider a firm with no debt, $X_t = 0$. Following a favorable cash-flow shock $dZ_t > 0$, the firm may retain the resulting earnings rather than distribute them, thereby accumulating cash ($X_{t+dt} > 0$). This cash buffer reduces the risk of having to issue costly equity after a subsequent adverse shock $dZ_{t+dt} < 0$. A higher ν increases this risk and therefore strengthens the firm's precautionary saving motive.

In contrast, when $\nu = 0$, the firm never holds cash: $X_t \geq 0$. This is because with $\nu = 0$, operating cash flow over $(t, t + dt)$ is $Y_{t-}dt$, which is locally deterministic. At $X_t = 0$, the firm has no interest expense and thus earns an after-tax profit of $(1 - \tau)Y_{t-}dt > 0$ with probability one. Since there is no contemporaneous cash-flow risk, the firm has no motive to retain this profit as cash. Moreover, since the firm's interest income from savings is taxed, the firm prefers to distribute $(1 - \tau)Y_{t-}dt$ to shareholders when $X_t = 0$ to avoid being taxed on its interest income. This gives rise to a nonnegative debt-EBIT ratio target: $\underline{x} \geq 0$.

The preceding discussion distinguishes the role of ν from that of the diffusion volatility σ . A positive ν directly exposes contemporaneous operating cash flow to first-order risk, thereby generating a precautionary demand for cash. In contrast, a higher σ affects the next instant's expected EBIT, $Y_{(t+dt)-}$, but has no effect on cash flow over $(t, t + dt)$. Since debt is continuously rolled over, the firm can immediately adjust its debt balance to the new EBIT level. Thus, a higher σ alone does not induce the firm to hold cash.²⁷

²⁷ Strengthening the downward jump component has a similar effect as increasing σ in a setting without a contemporaneous cash-flow shock ($\nu = 0$). A jump changes the debt-EBIT ratio from x to x/Z and thus increases leverage when $x > 0$. Thus, stronger or more frequent downward jumps can reduce the firm's desired leverage and increase the value of financial flexibility, but they do not generate an instantaneous funding shortfall at $x = 0$. Mathematically, when $x = 0$, a jump still leaves the firm's debt-EBIT ratio x/Z at zero.

Suppose now that $\nu > 0$ and consider the region in which the firm neither pays dividends nor raises external equity: $dU_t = dN_t = 0$. In this region, the firm's equity value, $P(Y, X)$, satisfies the following HJB equation:

$$\begin{aligned} rP(Y, X) = & \mu Y P_Y(Y, X) + (C + \Theta - Y) P_X(Y, X) + \frac{\sigma^2 Y^2}{2} P_{YY}(Y, X) \\ & + \frac{\nu^2 Y^2}{2} P_{XX}(Y, X) - \rho \sigma \nu Y^2 P_{XY}(Y, X) + \lambda \mathbb{E} [P(ZY, X) - P(Y, X)] . \end{aligned} \quad (30)$$

Compared with the baseline model with $\nu = 0$, two new terms involving P_{XX} and P_{XY} appear in the HJB equation (30). Economically, these two terms reflect the firm's precautionary saving demand against the *instantaneous* EBIT shock. Mathematically, when $\nu > 0$, the PDE (30) for the firm's equity value is an elliptic equation. In contrast, when $\nu = 0$, the PDE (30) reduces to a parabolic equation, since debt dynamics enter only through the first-order term P_X . The properties of elliptic equations differ substantially from those of parabolic equations.

Using the homogeneity property to simplify (30), we obtain the following ODE for the scaled equity value $p(x)$:

$$\begin{aligned} (r - \mu) p(x) = & [c(x) + \theta(c(x)) - 1 - \mu x] p'(x) + \frac{(\sigma^2 x^2 + 2\rho\nu\sigma x + \nu^2)}{2} p''(x) \\ & + \lambda \mathbb{E} [Zp(x/Z) - p(x)] . \end{aligned} \quad (31)$$

Compared with (21) for the baseline model, (31) contains the additional term $(2\rho\nu\sigma x + \nu^2)p''(x)/2$, which induces the firm to save. Whether $\nu > 0$ or $\nu = 0$ has very different economic implications. When $\nu > 0$, (31) is a *second-order* ODE even at $x = 0$. By contrast, when $\nu = 0$, (31) degenerates to a *first-order* ODE at $x = 0$. This mathematical difference implies very different precautionary-saving incentives. When $\nu > 0$, the firm's precautionary saving demand, captured by the term $\frac{\nu^2 p''(0)}{2}$, is economically significant even when the firm has no debt. In contrast, when $\nu = 0$, this term is zero.

Figure 8 provides a numerical illustration showcasing the effect of a locally random earnings process.²⁸ This figure shows that the firm has an optimal *target cash-EBIT ratio* of 2.2 (see $\underline{x} = -2.2$ on the solid line) rather than an optimal target debt-EBIT ratio of $\underline{x} = 2.1$. Consistent with our preceding analysis, since EBIT is now locally risky, the firm delays payouts and is willing

²⁸ We set $\nu = 2$ and $\rho = 0$, and use the same values for all other parameters as in subsection 4.5.

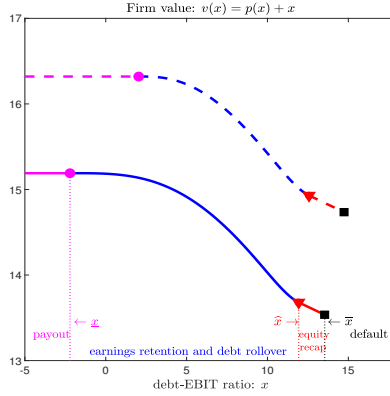


Figure 8: Firm value $v(x)$ in the extended model with both cash-holding and debt regions.

to save even in the presence of a tax disadvantage of holding cash. In summary, making earnings locally risky increases the firm’s precautionary saving demand, lowers firm value, and generates a cash-holding region.

We further extend the baseline model of Section 2 along two dimensions: hedging/risk taking and growth options, in Appendices I and J, respectively. When the firm hedges and/or seeks additional risk exposure by trading a risky asset, the optimal policy is region-dependent: at low leverage, where the firm is endogenously risk-averse, it eliminates its hedgeable risk exposure and retains only unhedgeable (idiosyncratic) risk; at high leverage, where the firm is endogenously risk-seeking, it takes on the maximum risk exposure allowed. With growth options, the firm’s exercise decision is non-monotonic in leverage: it exercises the growth option at both high and low leverage but bypasses it at intermediate leverage, reflecting the tension between value creation and prudent debt management, shaped by debt overhang (Myers, 1977) and risk shifting (Jensen and Meckling, 1976), the latter driven by the interaction between downward jumps and fixed equity issuance costs.

7 Conclusion

Inspired by the concluding remark in Fama and French (2005): “*it is best to regard the (pecking order and tradeoff) models as stable mates, with each having elements of truth that help explain some aspects of financing decisions,*” we develop a dynamic tradeoff model for a firm facing equity

issuance costs (Myers and Majluf, 1984) and risky cash flows with downward jump risk, to analyze the interconnected dynamics of leverage and credit risk.

The model predicts that default occurs suddenly, and equilibrium credit spreads are stochastic and sizable, even for bonds with very short maturity (consistent with the evidence in Duffie and Singleton, 2003). The model also predicts that the firm begins with a low target leverage and revolves its debt at equilibrium interest rates by letting leverage drift in response to realized earnings. When leverage jumps up in response to an earnings (jump) loss, the firm may proactively deleverage by issuing equity to manage its default risk.

The main predictions of our model are in line with the criteria summarized in the concluding remark in DeAngelo, Gonçalves, and Stulz (2018) for a credible capital structure theory: (1) most firms proactively deleverage; (2) financial flexibility and the option to borrow are valuable; (3) firms recognize the benefits of internally generated equity through earnings retention; and (4) there are significant interdependencies among leverage, cash balances, and decisions on retention versus payouts, driven by financial flexibility considerations. The model's predictions are also consistent with the survey findings of Graham and Harvey (2001) and Graham (2022).

We also explain why corporate leverage on average is low. Paradoxically, the explanation for the observed low corporate leverage is not that debt is costly, but that equity is costly. One would think that when equity is costly firms would want to rely more on debt. But that is a static intuition. From a dynamic perspective firms seek to delay or avoid costly equity issuance by maintaining financial slack and keeping leverage low today. When equity issuance is very costly, the firm barely takes on any debt.

Appendices

A Markov Perfect Equilibrium (MPE)²⁹: Definition

We begin by defining the firm's strategy set, which is the set of all Markov strategies. We then proceed to the definition of creditors' beliefs. Finally, we characterize the corporate debt and equity pricing formulae given creditors' beliefs over the firm's Markov strategies.

Definition 1. Markov strategies. Let S denote a triplet of corporate actions, consisting of the default timing $T^{\mathcal{D}}$, equity issuance N_t , and equity payout U_t :

$$S = (T^{\mathcal{D}}, \{N_t, U_t; t \in (0, T^{\mathcal{D}})\}).$$

Let $\mathcal{S}$ denote the set of Markov strategies. A strategy $S \in \mathcal{S}$ is Markov if 1.) it is adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ generated by $(\mathcal{B}_t, \mathcal{J}_t)_{t \geq 0}$; 2.) the process X_t , defined by equation (8), has a unique strong solution; 3.) $\mathbb{E}_t \left[\int_t^{T^{\mathcal{D}}} e^{-r(s-t)} dU_s \right] < \infty$, $\mathbb{E}_t \left[\int_t^{T^{\mathcal{D}}} e^{-r(s-t)} (dN_s + dH_s) \right] < \infty$; and 4.) there exist three sets, $\mathcal{D}$, $\mathcal{N}$, $\mathcal{U}$, in the space of $\Sigma = \{(Y, X) : Y > 0, X \geq 0\}$, with the following properties: a.)

$$T^{\mathcal{D}} = \inf\{t \geq 0 : (Y_t, X_t) \in \mathcal{D}\}, \quad (\text{A.1})$$

where $T^{\mathcal{D}} > t$ whenever $(Y_t, X_t) \notin \mathcal{D}$ and $X_t/Y_t > \ell$, and b.) for any $t < T^{\mathcal{D}}$,

$$N_t = \int_0^t \mathbf{1}_{(Y_s, X_{s-}) \in \mathcal{N}} dN_s, \quad U_t = \int_0^t \mathbf{1}_{(Y_s, X_{s-}) \in \mathcal{U}} dU_s, \quad (\text{A.2})$$

and $\Delta N_t := N_t - N_{t-} \geq 0$ and $\Delta U_t := U_t - U_{t-} \geq 0$ are functions of (Y_t, X_{t-}) .

Since dN_t and dU_t only depend on (Y_t, X_{t-}) , (8) implies that dX_t also only depends on (Y_t, X_{t-}) .

Creditors' beliefs. Let $\tilde{S}$ denote creditors' beliefs about the firm's strategy, which is also a triplet: $\tilde{S} = (T^{\tilde{\mathcal{D}}}, \{\tilde{N}_t, \tilde{U}_t; t \in (0, T^{\tilde{\mathcal{D}}})\})$. Given a Markov strategy S and creditors' belief $\tilde{S}$, the firm's equity value is given by:

$$\mathcal{P}(Y_t, X_t; S, \tilde{S}) = \mathbb{E}_t \left[\int_t^{T^{\mathcal{D}}} e^{-r(s-t)} (dU_s - dN_s - dH_s) \right], \quad (\text{A.3})$$

subject to the accounting identity (8) implied by S , and the credit spreads that depend on $\tilde{S}$. We next define the set of *feasible* Markov strategies, given creditors' belief $\tilde{S}$, which is a subset of $\mathcal{S}$, the set of Markov strategies. Let $\Phi(\tilde{S})$ denote the set of all feasible Markov strategies S for a given belief $\tilde{S}$.

Definition 2. Feasible Markov strategies. For a given belief $\tilde{S}$ with default region $\tilde{\mathcal{D}}$, a Markov strategy S is feasible, $S \in \Phi(\tilde{S})$, under the following conditions:

- (i) if there is no jump at t ($d\mathcal{J}_t = 0$ so that $Y_t = Y_{t-}$) and the firm does not default at $t < T^{\mathcal{D}}$, creditors believe that the firm is solvent at t : $(Y_t, X_{t-}) \notin \tilde{\mathcal{D}}$ and $(Y_t, X_t) \notin \tilde{\mathcal{D}}$; and if a jump arrives at t with $(Y_t, X_{t-}) \in \tilde{\mathcal{D}} \setminus \mathcal{D}$, then $(Y_t, X_t) \notin \mathcal{D}$;

²⁹ We use the term *Markov Perfect Equilibrium*, following standard usage in the dynamic capital-structure literature (see, e.g., DeMarzo and He (2021) and Malenko and Tsoy (2025)), to describe the firm's time-consistent, no-commitment optimization problem under sequentially rational Markov strategies. Our creditors are competitive and price-taking, so their behavior is fully pinned down by the zero-profit condition (A.4); no strategic interaction among creditors is involved. The equilibrium concept in this appendix therefore reduces to characterizing the firm's own sequentially rational, time-consistent policy.

(ii) *limited liability*: $\mathcal{P}(Y, X; S, \tilde{S}) \geq 0$ for all $(Y, X) \in \Sigma$.

Condition (i) requires that if the firm has been solvent up to time t , creditors expect it to continue to be solvent unless a jump arrives at t ($d\mathcal{J}_t = 1$). Moreover, conditional on being in the $\tilde{\mathcal{D}}$ set, the firm must either default or issue equity to immediately exit $\tilde{\mathcal{D}}$.

Next, we pin down the credit spread given creditors' belief $\tilde{S}$ and the firm's strategy $S \in \Phi(\tilde{S})$. A necessary condition for creditors to believe that the firm will default is a jump arrival. Therefore, for a small time interval dt , competitive debt pricing implies that the following zero-profit condition for creditors must hold:

$$X_{t-}(1 + rdt) = (X_{t-} + C_{t-}dt) \left[1 - \lambda \mathbb{E}_{t-} \left(\mathbf{1}_t^{\tilde{\mathcal{D}}} \right) dt \right] + \mathbb{E}_{t-} \left[\min\{L_t, X_{t-}\} \mathbf{1}_t^{\tilde{\mathcal{D}}} \right] \lambda dt, \quad (\text{A.4})$$

where $\mathbf{1}_t^{\tilde{\mathcal{D}}} = 1$ if $(Y_t, X_{t-}) \in \tilde{\mathcal{D}}$ and $\mathbf{1}_t^{\tilde{\mathcal{D}}} = 0$ otherwise. The first term on the right side of (A.4) equals the product of the total contractual repayment to creditors, $(X_{t-} + C_{t-}dt)$, over dt and the (risk-neutral) probability at time $t-$ that the firm won't default over dt : $\left[1 - \lambda \mathbb{E}_{t-} \left(\mathbf{1}_t^{\tilde{\mathcal{D}}} \right) dt \right]$. The second term gives the creditors' expected payoff if the firm defaults under the risk-neutral measure $\mathbb{Q}$. The left side of (A.4) states that creditors' expected rate of return over dt under $\mathbb{Q}$ must equal the risk-free rate r . Indeed, in equilibrium, the two sides of (A.4) must be equal (Duffie, 2001). Simplifying (A.4) yields:

$$C_{t-} = rX_{t-} + \lambda \mathbb{E}_{t-} \left[(X_{t-} - \min\{L_t, X_{t-}\}) \mathbf{1}_t^{\tilde{\mathcal{D}}} \right] = rX_{t-} + \lambda \mathbb{E}_{t-} \left[(X_{t-} - L_t)^+ \mathbf{1}_t^{\tilde{\mathcal{D}}} \right], \quad (\text{A.5})$$

where $x^+ = \max\{x, 0\}$. Using $C_{t-} = (r + \eta_{t-})X_{t-}$ given in (3) to rewrite (A.5), we obtain³⁰

$$\eta_{t-} = \lambda \mathbb{E}_{t-} \left[\left(1 - \frac{L_t}{X_{t-}} \right)^+ \mathbf{1}_t^{\tilde{\mathcal{D}}} \right]. \quad (\text{A.6})$$

Specifically, if $\tilde{\mathcal{D}} = \{(Y, X) : X/Y > \bar{x}\}$ for some constant $\bar{x} > 0$, then $\eta_{t-} = \eta(x_{t-}; \bar{x})$, where

$$\eta(x; \bar{x}) = \lambda \int_0^{x/\bar{x}} \left(1 - \frac{\ell Z}{x} \right)^+ dF(Z), \quad x \leq \bar{x}. \quad (\text{A.7})$$

Definition 3. Markov perfect equilibrium (MPE). *In an MPE, a feasible Markov strategy $S^* \in \Phi(S^*)$ satisfies the following best-response condition for all $(Y, X) \in \Sigma$:*

$$\mathcal{P}(Y, X; S^*, S^*) \geq \mathcal{P}(Y, X; S, S^*), \quad \forall S \in \Phi(S^*). \quad (\text{A.8})$$

Note that, in Definition 3, the credit spreads appearing on both sides of (A.8) are identical and are given by (A.6), where $\tilde{\mathcal{D}}$ is the default region under the strategy S^* .

B Derivation of the Solution and Summary of Main Results for the Baseline Model in Sections 2 and 4

Here we heuristically solve the baseline model using dynamic programming and then formally state the optimality results. Proofs of these results can be found in Internet Appendix IA.

³⁰ Since $\mathcal{D} = \tilde{\mathcal{D}}$ and $X_{T^{\mathcal{D}}-} \geq L_{T^{\mathcal{D}}}$ in an equilibrium, then we obtain (4).

B.1 Heuristic Derivation of Main Results in Section 4

We first solve the firm's dynamic optimization problem for a given credit spread function $\eta(x)$ and then use the competitive market pricing condition to determine the equilibrium $\eta(x)$ via a fixed-point argument. First, we characterize the optimal debt target $\underline{X}(Y)$ and payout region: $X < \underline{X}(Y)$. Using $P(Y, X) = P(Y, \underline{X}(Y)) + \underline{X}(Y) - X$, we have $-P_X(Y, \underline{X}(Y)) = 1$, which is the smooth-pasting condition.³¹ Since $\underline{X}(Y)$ is optimally chosen, the super-contact condition must hold, so that: $P_{XX}(Y, \underline{X}(Y)) = 0$ (Dumas, 1991). Using the homogeneity property to simplify these three equations, we obtain (12), (13), and (14).

Second, in the region where $\underline{X}(Y) \leq X < \hat{X}(Y)$, the firm retains its earnings and issues debt. Equity value $P(Y, X)$ satisfies:

$$rP(Y, X) = (C + \Theta - Y)P_X + \mu Y P_Y + \frac{\sigma^2 Y^2}{2} P_{YY} + \lambda \mathbb{E}[P(ZY, X) - P(Y, X)], \quad (\text{B.1})$$

where the $\mathbb{E}[\cdot]$ operator in (B.1) is evaluated with respect to the distribution $F(\cdot)$ for the recovery fraction Z . The last term captures the jump effect. When a jump arrives at t : $d\mathcal{J}_t = 1$, EBIT falls from Y_{t-} to $Y_t = ZY_{t-}$. Using the homogeneity property, we can simplify (B.1) to the integro-differential equation (21) for $p(x)$ in the region where $\underline{x} \leq x < \hat{x}$.

Third, we analyze the equity-issuance region: $\hat{X}(Y) \leq X \leq \bar{X}(Y)$. When $\hat{X}(Y) < \bar{X}(Y)$, $P(Y, X) = P(Y, \hat{X}(Y)) - [h_0 Y + (1 + h_1)(X - \hat{X}(Y))]$ holds. As the net equity issuance $M = X - \hat{X}(Y)$ is chosen optimally, $-P_X(Y, X) = -P_X(Y, \hat{X}(Y)) = (1 + h_1)$ must hold.³² Using the homogeneity property, we obtain (23) and (24). Finally, when $X > \bar{X}(Y)$, the firm defaults and $P(Y, X) = 0$, which implies (26).

In the next subsection, we state assumptions and establish key results.

B.2 Existence of Markov Perfect Equilibrium (MPE) and Verification Theorem

Assumption 1. (i) The expected growth rate of the EBIT process under the risk-neutral measure $\mathbb{Q}$, g given in (2), is strictly smaller than the risk-free rate: $g < r$. (ii) Equity value with zero leverage is larger than the liquidation recovery value: $\frac{1-\tau}{r-g} > \ell > 0$.

Part (i) of Assumption 1 is to ensure that equity value is finite and part (ii) of Assumption 1 requires that the firm's value under all equity financing is larger than its liquidation value.

Lemma 1. Under Assumption 1, in an MPE $S^* = (T^{\mathcal{D}}, \{N_t, U_t; t \in (0, T^{\mathcal{D}})\})$, where $T^{\mathcal{D}} = \inf\{t \geq 0 : X_t/Y_t \in \mathbb{X}^{\mathcal{D}}\}$ for some open set $\mathbb{X}^{\mathcal{D}} \subseteq [\ell, \infty)$, it must be the case that $\mathbb{X}^{\mathcal{D}} = (\bar{x}, \infty)$ for some $\bar{x} \geq \ell$.

This lemma establishes that default is characterized by a single connected region: $x > \bar{x}$.

Let $\mathcal{S}_x$ denote the set of Markov strategies $S = (T^{\mathcal{D}}, \{N_t, U_t; t \in (0, T^{\mathcal{D}})\})$ with $T^{\mathcal{D}} = \inf\{t \geq 0 : X_t/Y_t > \bar{x}\}$ for some $\bar{x} \geq \ell$. According to Lemma 1, we only need to study MPE $S^* \in \mathcal{S}_x$. We first establish the homogeneity property given creditors' beliefs that the firm's default policy is given by a cutoff threshold $\bar{x}$.

Proposition 1. Suppose Assumption 1 holds. Given creditors' belief $\tilde{S} \in \mathcal{S}_x$, let $S^* \in \mathcal{S}_x$ denote a feasible Markov strategy that satisfies $\mathcal{P}(Y, X; S^*, \tilde{S}) \geq \mathcal{P}(Y, X; \hat{S}, \tilde{S})$ for all $(Y, X) \in \Sigma$ and $\hat{S} \in \Phi(\tilde{S})$. Then the firm's equity value, $\tilde{P}(Y, X) = \mathcal{P}(Y, X; S^*, \tilde{S})$, satisfies the following homogeneous property:

$$\tilde{P}(Y, X) = Y \tilde{P}(1, X/Y). \quad (\text{B.2})$$

³¹ Note that the right side of the value-continuity condition $P(Y, X) - P(Y, \underline{X}(Y)) = \underline{X}(Y) - X$ is the lump-sum payment to shareholders. Let $\Delta = \underline{X}(Y) - X$. Dividing both sides by Δ and letting $\Delta \rightarrow 0$, we obtain the smooth-pasting condition.

³² As $P(Y, X)$ is continuous in X , $P(Y, \hat{X}(Y)+) = P(Y, \hat{X}(Y)-)$ where $\hat{X}(Y)+$ and $\hat{X}(Y)-$ are the right and left limits of $\hat{X}(Y)$, respectively. The homogeneity property then implies $p(\hat{x}+) = p(\hat{x}-)$.

Let $\tilde{p}(x) = \tilde{P}(1, x)$ and $v = \inf\{x > 0 : \tilde{p}(x) = 0\}$. Then $\tilde{p}(x) = 0$ for any $x \geq v$ and $\mathcal{D} = \{(Y, X) : X/Y > v\}$ is the default region under the strategy S^* .

Assumption 2. For any creditors' belief $\tilde{S} \in \mathcal{S}_x$, there exists an optimal Markov strategy $S \in \Phi(\tilde{S})$, such that $\mathcal{P}(Y, X; S, \tilde{S}) \geq \mathcal{P}(Y, X; \hat{S}, \tilde{S})$ for all $(Y, X) \in \Sigma$ and $\hat{S} \in \Phi(\tilde{S})$.

Intuitively speaking, given creditors' beliefs that the firm's default policy is given by a cutoff threshold $\bar{x}$, Assumption 2 requires that the firm's optimization problem has a solution. Under these assumptions, we establish the existence of an MPE that features a cutoff default threshold.

Proposition 2. Under Assumptions 1-2, there exists an MPE in which the default region is $\mathcal{D} = \{(Y, X) : X/Y > \bar{x}\}$ for some $\bar{x} \geq \ell$. In equilibrium, the firm's equity value $P(Y, X)$ is strictly increasing in Y and strictly decreasing in X for any $Y > 0$ and $X \in [0, \bar{x}Y]$.

Let $p(x; S^*)$ denote the scaled equity value associated with a candidate equilibrium strategy S^* : $p(x; S^*) = \mathcal{P}(1, x; S^*, S^*)$.

Next, we state a verification theorem for an MPE.

Theorem 1. Verification theorem. Consider a Markov strategy $S^* \in \mathcal{S}_x$ that satisfies (B.2) and includes a default region $\mathcal{D}^* = \{(Y, X) : X/Y > \bar{x}\}$ for some $\bar{x} \geq \ell$. Let $v = \inf\{x > 0 : p(x; S^*) = 0\}$. S^* is an MPE if 1.) $v = \bar{x}$, 2.) the function $P(Y, X) = \mathcal{P}(Y, X; S^*, S^*)$ satisfies $P(Y, X) = 0$ for any $X \geq \bar{x}Y > 0$, and 3.) the following variational inequality for $(Y, X) \in \mathbb{Z} = \{(Y, X) | 0 < Y < \infty, 0 \leq X \leq \bar{x}Y\}$ holds:

$$\max\{\mathcal{L}P, P_X(Y, X) + 1, P^N(Y, X) - P(Y, X)\} = 0. \quad (\text{B.3})$$

In (B.3), $P^N(Y, X)$ is defined by

$$P^N(Y, X) = \sup_{M > 0} P(Y, X - M) - h_0 Y - (1 + h_1)M \quad (\text{B.4})$$

and $\mathcal{L}P$ is given by

$$\mathcal{L}P = \mu Y P_Y + (C + \Theta - Y) P_X + \frac{\sigma^2 Y^2}{2} P_{YY} + \lambda \mathbb{E}[P(ZY, X) - P(Y, X)] - rP, \quad (\text{B.5})$$

where $\Theta = \theta(c)Y$ is the firm's tax payment (with $\theta(c)$ given in (5) and $c = C/Y$), $C = (r + \eta(X/Y; \bar{x}))X$ is the interest payment, and $\eta(x; \bar{x})$ is the credit spread given in (A.7).

Next, we prove the existence of the best MPE, which is the equilibrium with the highest equity value.

B.3 The Best MPE

Theorem 2. Existence of the best MPE. Under Assumptions 1-2, there exists an MPE associated with strategy $S^* \in \mathcal{S}_x$ that satisfies $\mathcal{P}(Y, X; S^*, S^*) \geq \mathcal{P}(Y, X; S, S)$ for all $(Y, X) \in \Sigma$ and any MPE associated with strategy $S \in \mathcal{S}_x$, where $\mathcal{S}_x$ denotes the set of Markov strategies S such that $T^{\mathcal{D}} = \inf\{t \geq 0 : X_t/Y_t > \bar{x}\}$ for some $\bar{x} \geq \ell$.

Theorem 3. Solving for the best MPE. Under Assumptions 1-2, for any MPE $S^* \in \mathcal{S}_x$, the firm's scaled equity value $p(x) = p(x; S^*)$ is the unique continuous and bounded viscosity solution to the following variational inequality.³³

$$\max\{\mathcal{L}^{\bar{x}}p(x), 1 + p_x(x), p^N(x) - p(x)\} = 0, \quad x \in [0, \bar{x}], \quad (\text{B.6})$$

³³ Boundary conditions are $p_x(0) = -1$ and $p(\bar{x}) = 0$. See Barles and Souganidis (1991) and Seydel (2009) for the technical details for the viscosity solution. When $\bar{x} > \ell$, we additionally require that $p^N(\bar{x}) = 0$.

where $\bar{x} = \inf\{x > 0 : p(x) = 0\} \geq \ell$ and

$$\begin{aligned} \mathcal{L}^{\bar{x}}p(x) &= [c(x; \bar{x}) + \theta(c(x; \bar{x})) - 1 - \mu x] p_x(x) \\ &\quad + \frac{\sigma^2 x^2}{2} p_{xx}(x) + \lambda \mathbb{E}[Zp(x/Z) - p(x)] - (r - \mu)p(x), \end{aligned} \quad (\text{B.7})$$

and $p(x) = 0$ for any $x \geq \bar{x}$. In (B.7), $c(x; \bar{x}) = (r + \eta(x; \bar{x}))x$ is the interest payment, $\eta(x; \bar{x})$ is the credit spread given in (A.7), and $p^N(x) = \max_{m>0} p(x - m) - h_0 - (1 + h_1)m$. Denote $\bar{\mathbb{X}}$ as the set of default boundaries $\bar{x}$ for all MPEs in the set $\mathcal{S}_x$ and $\bar{x}^* = \sup \bar{\mathbb{X}}$. Then $\bar{x}^* \leq \frac{1}{r-g}$, and $\bar{x}^* \in \bar{\mathbb{X}}$ is the default boundary of the best equilibrium in the set $\mathcal{S}_x$.

Using Theorem 3, to solve for the best equilibrium, we only need to find the maximum trigger $\bar{x} \in [\ell, \frac{1}{r-g}]$, such that (B.6) admits a solution $p(x)$ satisfying $p(x) > 0$ for all $x < \bar{x}$.

B.4 Four-region Solution Characterization of MPE

Below, we show that with costly external equity (when $h_0 > 0$ or $h_1 > 0$), each of the four regions in an MPE—the payout region, the earnings-retention and debt-rollover region, the equity-issuance region, and the default region—is characterized by a single connected interval. We state the main results and relegate the details of the proof for the four-region solution to Internet Appendix IA. For the case of costless equity issuance (when $h_0 = h_1 = 0$), a characterization of the MPE is provided in Internet Appendix IB.

First, we show that the payout policy features a single connected region. To prove this result, we impose the following technical condition.

Condition 1. $(1 - \tau)(1 - \frac{\ell}{x})zF'(z) - \tau F(z)$ is strictly increasing in $z \in [0, 1]$.

Condition 1 is needed to establish the monotonicity of $\mathcal{L}^{\bar{x}}p(x)$ in the payout region. This monotonicity is crucial for showing that $\mathcal{L}^{\bar{x}}p(x) \leq 0$, and hence for proving the suboptimality of the “earnings retention and debt rollover” policy in the payout region. For the power law $F(z) = z^\beta$ used in our quantitative analysis, Condition 1 is equivalent to $(1 - \tau)(1 - \frac{\ell}{x})\beta - \tau > 0$. This condition is satisfied in all cases considered in the paper, including the comparative dynamics in Section 5; see Tables 3-8.

Proposition 3. Under Assumptions 1-2, in an MPE $S^* \in \mathcal{S}_x$ satisfying Condition 1, the payout policy is characterized by a single connected region: $x < \underline{x}$ for a threshold $\underline{x} < \bar{x}$.

To prove that the payout policy is characterized by the single connected region stated in Proposition 3, a key step is to show that, once the highest payout boundary is identified, a linear extension of the equity value to all lower values of x still satisfies the variational inequality (B.6), implying that all lower-leverage states must lie in the payout region; see Internet Appendix IA for details.

Next, we establish that equity financing is also characterized by a single connected region. To do so, we impose the following condition at $\underline{x}$.

Condition 2. $\mathcal{L}^{\bar{x}}(\bar{x} - x)^+ > 0$ for $x = \underline{x}$.

This condition states that, at the payout boundary $\underline{x}$, the policy of paying out $\bar{x} - \underline{x}$ and then immediately defaulting is suboptimal. Condition 2 is equivalent to $A(\underline{x}) - \lambda \int_{\underline{x}/\bar{x}}^1 (z\bar{x} - \underline{x})dF(z) + (r - \mu + \lambda)(\bar{x} - \underline{x}) < 0$, where $A(x) = c(x; \bar{x}) + \theta(c(x; \bar{x})) - 1 - \mu x$. We can verify that this condition is satisfied in all cases considered in the paper.

Proposition 4. Under Assumptions 1-2, in an MPE $S^* \in \mathcal{S}_x$ satisfying Conditions 1-2, equity financing is characterized by an interval $[\hat{x}, \bar{x}]$, where $\hat{x} \in (\underline{x}, \bar{x}]$ and $\underline{x}$ is the payout boundary given in Proposition 3.

Finally, we summarize the four-region solution in the following theorem.

Theorem 4. *Under Assumptions 1–2, in an MPE $S^* \in \mathcal{S}_x$ satisfying Conditions 1–2, there exist thresholds $\underline{x} < \hat{x} \leq \bar{x}$ such that: i) in the region $x < \underline{x}$, the firm makes an equity payout, and $p(x)$ is given by (12); ii) in the region $x \in [\underline{x}, \hat{x})$, the firm relies on earnings retention and debt financing and $p(x)$ satisfies the integro-differential equation (21); iii) in the region $x \in [\hat{x}, \bar{x}]$, the firm issues equity and $p(x)$ is given by (23); and iv) in the region $x > \bar{x}$, the firm defaults and $p(x) = 0$.*

We provide proofs of Lemma 1, Propositions 1–4 and Theorems 1–4 in Internet Appendix IA.

C Long-Term Lending Contract Implementation

This appendix formulates the firm’s and the lender’s problems under the long-term lending contract described in subsection 4.6.

The lender’s valuation and participation condition. Let $S = (T^{\mathcal{D}}, \{N_t, U_t : t \in (0, T^{\mathcal{D}})\})$ denote the firm’s strategy under the agreement, and let $D_{t-}(S)$ denote the lender’s continuation value immediately before time t , conditional on the firm being solvent. Given the firm’s strategy S and the state-contingent interest rate, the lender’s continuation value at time $t-$ is

$$D_{t-}(S) = \mathbb{E}_{t-} \left[\int_{[t, T^{\mathcal{D}})} e^{-r(s-t)} (C_{s-} ds - dX_s) + e^{-r(T^{\mathcal{D}}-t)} \min\{X_{T^{\mathcal{D}}-}, L_{T^{\mathcal{D}}}\} \right], \quad (\text{C.1})$$

where X satisfies the accounting identity (8) implied by S . The lender’s contractual cash flows have three components. First, the lender receives the interest payment $C_{s-} ds$. Second, $-dX_s$ accounts for changes in the lender’s funding exposure. If the firm draws an additional amount, $dX_s > 0$, the lender makes a cash advance. If the firm reduces its balance, $dX_s < 0$, the lender receives a repayment. Third, following default, the lender receives the contractual recovery value $\min\{X_{T^{\mathcal{D}}-}, L_{T^{\mathcal{D}}}\}$. Because the lender commits at $t = 0$ to advance X_0 at contract inception, the following nonnegative profit condition must hold:

$$D_0(S) \geq X_0. \quad (\text{C.2})$$

The firm’s optimization problem. Taking the long-term lending contract as given, the firm chooses its default, equity-issuance, and payout policies $S = (T^{\mathcal{D}}, \{N_t, U_t : t \in (0, T^{\mathcal{D}})\})$ to maximize its equity value at time 0:

$$\mathcal{P}(Y_0, X_0; S) = \mathbb{E} \left[\int_0^{T^{\mathcal{D}}} e^{-rs} (dU_s - dN_s - dH_s) \right] \quad (\text{C.3})$$

subject to the accounting identity (8) implied by S . Importantly, when evaluating a deviation from a candidate policy, the firm takes the contractual schedule $\eta(x; \bar{x})$ as fixed, as the lender commits to the long-term lending contract at $t = 0$ and does not reprice the contract in response to the firm’s deviation.

Proposition 5. *Let $S^* = (T^{\mathcal{D}*}, \{N_t^*, U_t^* : t \in (0, T^{\mathcal{D}*})\})$ denote an MPE of the baseline model, as defined in Definition 3, with $T^{\mathcal{D}*} = \inf\{t \geq 0 : X_t/Y_t > \bar{x}\}$ for some $\bar{x} \geq \ell$. Consider the exclusive long-term lending contract in which the state-contingent spread schedule $\eta(x; \bar{x})$ is fixed as in (A.7).*

The firm’s optimal policy under this long-term contract is the same as the strategy S^ in the MPE of the baseline model. The lender’s continuation value, $D_{t-}(S^*)$, equals the firm’s debt balance at all t prior to default: $D_{t-}(S^*) = X_{t-}$ whenever $x_{t-} < \bar{x}$, and the lender breaks even at $t = 0$.*

Therefore, the equilibrium outcome under the short-term debt formulation can be implemented through the constructed long-term lending contract. Finally, neither the lender nor the borrower has incentives to renegotiate the long-term contract at any t .

D Calibration, Simulation and Supporting Details for Section 5

D.1 Change of Measure

In this appendix, we derive the EBIT process $\{Y_t\}$ under the risk-neutral measure $\mathbb{Q}$ starting from its dynamics under the physical measure $\mathbb{P}$ (see Duffie, 2001, for a textbook treatment). Fix a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and consider the following EBIT process:

$$\frac{dY_t}{Y_{t-}} = \mu^{\mathbb{P}} dt + \sigma d\mathcal{B}_t^{\mathbb{P}} - (1 - Z)d\mathcal{J}_t^{\mathbb{P}}, \quad (\text{D.1})$$

where $\mu^{\mathbb{P}}$ is the drift parameter, $\mathcal{B}^{\mathbb{P}}$ is a standard Brownian motion, and $\mathcal{J}^{\mathbb{P}}$ is a pure jump process with a constant arrival rate $\lambda^{\mathbb{P}}$. Assume that the stochastic discount factor $\mathbb{M}_t$ evolves (under $\mathbb{P}$) as follows:

$$\frac{d\mathbb{M}_t}{\mathbb{M}_{t-}} = -r dt - \vartheta d\mathcal{Z}_t^{\mathbb{P}} - (1 - e^{\kappa})(d\mathcal{J}_t^{\mathbb{P}} - \lambda^{\mathbb{P}} dt), \quad \mathbb{M}_0 = 1, \quad (\text{D.2})$$

where $\mathcal{Z}_t^{\mathbb{P}}$ is a Brownian motion driving the systematic component of diffusion risk, the parameter ϑ measures the market price of this risk, and the parameter κ captures the market price of jump risk. Let ρ denote the (instantaneous) correlation coefficient between the firm's EBIT diffusion shock $d\mathcal{B}_t^{\mathbb{P}}$ and the systematic diffusion shock $d\mathcal{Z}_t^{\mathbb{P}}$.

Next, define the risk-neutral measure $\mathbb{Q}$ as the probability measure under which, for any $\mathcal{F}_T$ -measurable payoff A_T , the following equation holds for all $t \in [0, T]$:

$$\mathbb{E}_t^{\mathbb{P}} \left[\frac{\mathbb{M}_T}{\mathbb{M}_t} A_T \right] = \mathbb{E}_t[e^{-r(T-t)} A_T], \quad (\text{D.3})$$

where $\mathbb{E}_t^{\mathbb{P}}[\cdot] = \mathbb{E}^{\mathbb{P}}[\cdot | \mathcal{F}_t]$ is the expectation operator under the physical measure $\mathbb{P}$, and $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t]$ is the expectation operator under the risk-neutral measure $\mathbb{Q}$. Equivalently, $\mathbb{Q}$ is characterized by the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathbb{M}_t e^{rt}$, which holds at all $t > 0$.

To ease exposition, we assume that the recovery distribution of Z is the same under the physical measure $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$, so that the jump risk premium arises only from uncertainty about the jump arrival time. By Girsanov's theorem, the firm's risk-adjusted EBIT process under the risk-neutral measure $\mathbb{Q}$ is given by equation (1), with $\lambda = e^{\kappa} \lambda^{\mathbb{P}}$, $\mu = \mu^{\mathbb{P}} - \rho \vartheta \sigma$, and $d\mathcal{B}_t = d\mathcal{B}_t^{\mathbb{P}} + \rho \vartheta dt$. Note that $\kappa = \ln(\lambda/\lambda^{\mathbb{P}})$ measures the jump risk premium.

D.2 Parameter Choices

We set the annual risk-free rate r at 6% and the (expected) EBIT growth rate g to 1% per annum under the risk-neutral measure as in Leland (1994a), Goldstein, Ju, and Leland (2001), and other contingent-claims capital-structure models. We set the corporate tax rate τ at the current US federal rate of 21%. We leave aside personal taxation and other tax code complications for simplicity.³⁴

We assume that the cumulative distribution function $F(z)$ for the recovery fraction Z follows a power law (Luttmer, 2007; Gabaix, 2009): $F(z) = z^{\beta}$ for $0 \leq z \leq 1$. The lower is the exponent β , the more fat-tailed is the distribution of $(1 - Z)$.

We calibrate the diffusion-volatility parameter σ , the jump-arrival rate $\lambda^{\mathbb{P}}$, and the distribution parameter β for the recovery fraction Z by matching the variance, skewness, and excess kurtosis of the

³⁴ Hu (2023) incorporates a differential tax treatment of debt versus equity in a setting similar to DeMarzo and He (2021).

annual logarithmic EBIT growth as follows. Using (D.1), the annual logarithmic EBIT growth rate is

$$y \equiv \log \left(\frac{Y_{t+1}}{Y_t} \right) = \mu^{\mathbb{P}} - \frac{\sigma^2}{2} + \sigma \left(\mathcal{B}_{t+1}^{\mathbb{P}} - \mathcal{B}_t^{\mathbb{P}} \right) + \sum_{i=1}^{N_Z} \log Z_i, \quad (\text{D.4})$$

where $N_Z \sim \text{Poisson}(\lambda^{\mathbb{P}})$ is the number of jumps over the one-year interval $(t, t+1]$, and $\{Z_i\}_{i \geq 1}$ are i.i.d. random variables with cumulative distribution function $F(z) = z^\beta$ for $z \in [0, 1]$. By assumption, N_Z , $\{Z_i\}_{i \geq 1}$, and $\mathcal{B}_{t+1}^{\mathbb{P}} - \mathcal{B}_t^{\mathbb{P}}$ are independent.

Under $F(z) = z^\beta$, the random variable $-\log Z_i$ is exponentially distributed with rate β , implying that $\mathbb{E}[(\log Z_i)^n] = (-1)^n \frac{n!}{\beta^n}$. It follows that the model-implied variance, skewness, and excess kurtosis of the annual logarithmic EBIT growth rate in (D.4) are

$$\text{Var}(y) = \sigma^2 + \frac{2\lambda^{\mathbb{P}}}{\beta^2}, \quad \text{Skew}(y) = \frac{-6\lambda^{\mathbb{P}}/\beta^3}{[\text{Var}(y)]^{3/2}}, \quad \text{ExcessKurt}(y) = \frac{24\lambda^{\mathbb{P}}/\beta^4}{[\text{Var}(y)]^2}. \quad (\text{D.5})$$

We jointly match these three model-implied moments to the corresponding Compustat estimates of 0.18, -0.08 , and 0.09 , respectively.³⁵ Solving (D.5), we then obtain (approximately) the following parameter values: $\sigma = 40.6\%$, $\lambda^{\mathbb{P}} = 0.8$, and $\beta = 9$.

We calibrate λ by targeting a risk-adjusted expected proportional EBIT loss from jumps of $\lambda(1 - \mathbb{E}(Z)) = \lambda/(\beta + 1) = 12\%$, in line with estimates in the credit-risk literature (Chen, 2010; Huang and Huang, 2012). With $\beta = 9$, the expected proportional EBIT loss per jump is $1/(\beta + 1) = 10\%$, so this target implies a risk-neutral jump-arrival rate of $\lambda = 1.2$ per year. The resulting risk-neutral intensity is 1.5 times the physical intensity, $\lambda^{\mathbb{P}} = 0.8$, calibrated above. Using (2), $g = 1\%$, and $\lambda(1 - \mathbb{E}(Z)) = 12\%$, we obtain $\mu = g + \lambda(1 - \mathbb{E}(Z)) = 13\%$. For simplicity, we set ρ , the correlation coefficient between the firm's EBIT diffusion shock $d\mathcal{B}_t^{\mathbb{P}}$ and the systematic diffusion shock $dZ_t^{\mathbb{P}}$, to zero. This implies that $\mu^{\mathbb{P}} = \mu = 13\%$.³⁶

As in Huang and Huang (2012), we target a recovery value at default equal to 51% of the debt face value. This yields a liquidation recovery parameter of $\ell = 5.35$, which lies within the range of estimates in the literature (Lian and Ma, 2021).

D.3 Distribution and Moments Based on Simulation

We solve the firm's optimization problem under the risk-neutral measure $\mathbb{Q}$, as the discount rate under $\mathbb{Q}$ is always the risk-free rate r , which is *constant*. After obtaining the equity value $p(x)$ and policy thresholds for the debt-EBIT ratio, $\underline{x}$, $\tilde{x}$, $\hat{x}$, and $\bar{x}$, we simulate corporate decisions, e.g., payout, equity recapitalization, and default, under the physical measure $\mathbb{P}$. We use the risk-neutral measure $\mathbb{Q}$ for valuation and optimization analysis for technical convenience. By contrast, we discuss corporate policies

³⁵ The dataset is Compustat North American Fundamentals for publicly traded firms in North America, which contains balance sheets reported annually by companies between 1970 and 2020. We drop firms with missing total assets, negative sales, negative cash and short-term investments, negative capital expenditures, negative debt in current liabilities, negative depreciation and amortization, negative R&D expenses, missing SIC, zero common shares outstanding, or negative common equity. We focus on firms incorporated in the United States. Financial firms (SIC code 6000-6999) and utility firms (SIC codes 4900-4999) are excluded from the sample. The final sample comprises 91,657 firm-year observations. We calculate the logarithmic EBIT growth rates and winsorize at 5% on both sides.

³⁶ Note that $\mu^{\mathbb{P}}$ does not affect the equilibrium solution, including $p(x)$ and the thresholds $\underline{x}$, $\tilde{x}$, $\hat{x}$, and $\bar{x}$, but it affects the simulation procedure in Appendix D.3. We also consider alternative values of $\mu^{\mathbb{P}}$ and find that the simulation moments are quantitatively insensitive to reasonable choices of $\mu^{\mathbb{P}}$. For example, across $\mu^{\mathbb{P}} \in [10\%, 20\%]$, the resulting average market leverage differs by less than 2.5%, and the resulting annual equity-issuance frequency differs by less than 2%.

and relate them to the empirical corporate finance literature, using statistics under the physical measure $\mathbb{P}$, because we are interested in corporate policies in the real world and empirical research is conducted under the physical measure. Specifically, we simulate 100,000 paths under $\mathbb{P}$. We set the time step to $\Delta t = 0.01$ years and define $t_n = n\Delta t$. For each path, $\mathbf{x}_n$ denotes the simulated x_{t_n} . Each simulated firm starts with no debt at birth and exits the economy when it defaults.

- (i) The entering firm is endowed with an operating technology and positive EBIT but has no debt. Immediately before making its initial financing decision, its debt level is $x_{0-} = 0$. The firm then issues debt costlessly and makes a payout, moving to the payout boundary. We therefore set $\mathbf{x}_0 = \underline{x}$.³⁷
- (ii) Given the debt-EBIT ratio $\mathbf{x}_n$ at time $t_n = n\Delta t < T^{\mathcal{D}}$, the debt-EBIT ratio at time t_{n+1} , prior to any decision by the firm, has the following approximation:

$$\mathbf{x}_{n+1}^{\text{prior}} = \mathbf{x}_n + \mu_x^{\mathbb{P}}(\mathbf{x}_n) \Delta t - \sigma \mathbf{x}_n \Delta \mathcal{B}_{t_n}^{\mathbb{P}} + \left(\frac{\mathbf{x}_n}{Z} - \mathbf{x}_n \right) \Delta \mathcal{J}_{t_n}^{\mathbb{P}},$$

where $\mu_x^{\mathbb{P}}(x_{t-}) = [c(x_{t-}) + \theta(c(x_{t-})) - 1] - (\mu^{\mathbb{P}} - \sigma^2) x_{t-}$, $\Delta \mathcal{B}_{t_n}^{\mathbb{P}} = \mathcal{B}_{t_{n+1}}^{\mathbb{P}} - \mathcal{B}_{t_n}^{\mathbb{P}}$ is normally distributed with mean zero and variance Δt , $\Delta \mathcal{J}_{t_n}^{\mathbb{P}} = 1$ with probability $\lambda^{\mathbb{P}} \Delta t$ and $\Delta \mathcal{J}_{t_n}^{\mathbb{P}} = 0$ with probability $1 - \lambda^{\mathbb{P}} \Delta t$, and Z follows a power law: $\mathbb{P}(Z \leq z) = z^{\beta}$ for $0 \leq z \leq 1$.

- (iii) If $\mathbf{x}_{n+1}^{\text{prior}} \leq \bar{x}$, the debt-EBIT ratio at time t_{n+1} after the firm's decision is given by

$$\begin{aligned} \mathbf{x}_{n+1} &= \underline{x}, & \text{if } \mathbf{x}_{n+1}^{\text{prior}} &\leq \underline{x}, \\ \mathbf{x}_{n+1} &= \mathbf{x}_{n+1}^{\text{prior}}, & \text{if } \underline{x} < \mathbf{x}_{n+1}^{\text{prior}} < \hat{x}, \\ \mathbf{x}_{n+1} &= \tilde{x}, & \text{if } \hat{x} \leq \mathbf{x}_{n+1}^{\text{prior}} \leq \bar{x}. \end{aligned}$$

- (iv) If $\mathbf{x}_{n+1}^{\text{prior}} > \bar{x}$, we set $\mathbf{x}_{n+1} = \mathbf{x}_{n+1}^{\text{prior}}$, the simulation stops, and the default time is set to t_{n+1} . Otherwise, steps (ii)–(iii) are repeated.

We denote by $\{\mathbf{x}_{j,n}\}_{0 \leq n \leq n_j^{\mathcal{D}}}$ the sample path in the j -th simulation, where $n_j^{\mathcal{D}} = \inf\{n : \mathbf{x}_{j,n} > \bar{x}\}$ is the default time index in the j -th simulation, so that the corresponding default time is $T_j^{\mathcal{D}} = \Delta t n_j^{\mathcal{D}}$. We construct a pooled empirical distribution using all simulated observations prior to default:

$$\hat{F}(x) = \frac{1}{\sum_{j=1}^{10^5} n_j^{\mathcal{D}}} \sum_{j=1}^{10^5} \sum_{n=0}^{n_j^{\mathcal{D}}-1} \mathbf{1}_{\mathbf{x}_{j,n} \leq x}. \quad (\text{D.6})$$

Note that the empirical distribution $\hat{F}(x)$ characterizes a firm's time spent in different debt-EBIT ratios prior to default.

We may view $\hat{F}(x)$ as the model counterpart of the pooled firm-year distribution in COMPUSTAT. In COMPUSTAT, summary statistics such as the mean, median, and percentiles of leverage are typically computed by pooling observations across firms and years, so that each firm-year observation receives equal weight. Our pooled empirical distribution follows the same logic: states that are visited more frequently, or for longer durations, receive larger weight through the time occupation measure. This

³⁷ The model-implied moments are quantitatively insensitive to reasonable alternative entrant states. For alternative initial states $\mathbf{x}_0 \in [\underline{x}, \tilde{x}]$, the resulting average market leverage differs from the baseline by less than 1% in levels, and the annual equity-issuance frequency differs by less than 0.1% in levels.

explains why it is appropriate to derive the statistics of market leverage using the empirical distribution $\hat{F}(x)$ given in (D.6).

Using the pooled empirical distribution $\hat{F}(x)$ defined in (D.6), the leverage statistics reported in the main text are computed as follows. The model-implied mean market leverage is given by $\int_0^{\bar{x}} \frac{x}{x+p(x)} d\hat{F}(x)$, and the α -quantile of market leverage is given by $\frac{x_\alpha}{x_\alpha+p(x_\alpha)}$, where $p(x)$ is the scaled equity value solved under $\mathbb{Q}$, and $x_\alpha = \inf\{x \geq 0 : \hat{F}(x) \geq \alpha\}$.

We can also interpret the pooled mean using a replacement economy in which each defaulting firm is immediately replaced by a new firm following the same entry and financing rules. In this replacement economy, long-run average market leverage is calculated by summing market leverage across all observation dates and dividing by the total number of observations. The simulated pooled average, $\int_0^{\bar{x}} \frac{x}{x+p(x)} d\hat{F}(x)$, estimates the long-run mean in this replacement economy.

Moreover, we compute the simulated equity-issuance frequency using the same firm-year definition as the SEO probability in DeAngelo, DeAngelo, and Stulz (2010). Specifically, for each simulated firm, we record annually whether the firm issues equity during the year. We then aggregate the number of firm-years with equity issuance across all simulated firms and divide it by the total number of simulated firm-years, where the latter is the sum of the number of years that each simulated firm remains alive before default. This construction corresponds to the fraction of firm-year observations in which equity issuance occurs.

D.4 Simulating Peak-to-Trough Deleveraging Episodes

To generate model predictions that can be related to the empirical findings on peak-to-trough deleveraging in DeAngelo, Gonçalves, and Stulz (2018), we conduct the following finite-horizon simulation exercises. We follow Steps (i)-(iii) in Appendix D.3 to generate a simulated path for each firm, using the parameter values in Appendix D.2 and the same four thresholds, $\underline{x}$, $\tilde{x}$, $\hat{x}$, and $\bar{x}$. For each simulated path, we terminate the path when either the firm defaults or the maximum simulation horizon T is reached. That is, let the termination time for simulated firm j be $T_j = \min\{T_j^{\mathcal{D}}, T\}$, where $T_j^{\mathcal{D}}$ denotes the default time of simulated firm j .

We use a finite horizon in this simulation because the goal here is to study corporate deleveraging from a firm's peak ML to its later trough, as in DeAngelo, Gonçalves, and Stulz (2018), who identify deleveraging episodes from finite COMPUSTAT histories. For Table 2, we set $T = 30$ years in the simulations.³⁸ In addition, to align the simulated sample with DeAngelo, Gonçalves, and Stulz (2018), we impose the same sample-selection criterion used in their main analysis: we retain only simulated firms with at least five years of post-peak data.

The underlying simulation time step remains $\Delta t = 0.01$ years, as in Appendix D.3. To match the annual firm-year data used by DeAngelo, Gonçalves, and Stulz (2018), we record observations only once per year. For each simulated firm j , we retain annual observations and denote its market leverage in year t by $ML_{j,t}$, $t = 0, 1, \dots, [T_j]$, where $[T_j]$ is the greatest integer less than or equal to T_j . We then identify the annual peak market leverage, $ML_j^{\text{peak}} = \max_{0 \leq t \leq [T_j]} ML_{j,t}$, and define t_j^{peak} as the first year in which this maximum is attained. The subsequent trough is $ML_j^{\text{trough}} = \min_{t_j^{\text{peak}} \leq t \leq [T_j]} ML_{j,t}$, and t_j^{trough} denotes the first year in which this minimum is attained. The peak-to-trough decline in ML is $\Delta ML_j = ML_j^{\text{trough}} - ML_j^{\text{peak}}$, and the duration of the deleveraging episode is $\text{Dur}_j = t_j^{\text{trough}} - t_j^{\text{peak}}$.

³⁸ As a robustness check, we repeat the simulation using alternative sample horizons $T \in [20, 40]$. Across these horizons, the median ML at the peak, the median ML at the post-peak trough, and the median peak-to-trough decline in ML each differ by less than 0.1 from their respective baseline values under $T = 30$. The median years from the ML peak to the trough differs by less than one year from its baseline value. Thus, the reported peak-to-trough results are robust to reasonable variations in the finite sample horizon.

Table 4: EFFECTS OF EQUITY ISSUANCE COSTS ON THRESHOLDS.

	$\underline{x}$	$\tilde{x}$	x^0	$\breve{x}$	$\hat{x}$	$\bar{x}$
baseline	2.1	5.3	8.9	10.4	12.6	14.7
$h_0 = 0$	3.1	10.4	9.1	10.4	10.4	15.6
$h_0 = 2$	1.5	3.5	8.5	9.8	12.3	13.2
$h_1 = 0$	3.0	3.0	9.1	10.8	13.3	15.9
$h_1 = 0.3$	1.5	6.0	8.4	9.6	11.4	12.9

The second column of Table 2 reports the cross-firm medians of ML_j^{peak} , ML_j^{trough} , ΔML_j , and Dur_j among the simulated firms.

E Solution Algorithm

We numerically solve the integro-differential equation for the equity value function, $p(x)$, given in (21) using the following algorithm for the baseline model in which the equity issuance region $x \in (\hat{x}, \bar{x})$ exists.

1. Start with a sufficiently large region $(x_{\min}, x_{\max})$ by setting $x_{\min} = 0$ and $x_{\max} = v^*$, where v^* is the firm's equity value under the costless equity issuance case. We use the superscript (i) to denote the values of $p(x)$, $\underline{x}$, $\hat{x}$ in the i^{th} iteration: $p^{(i)}(x)$, $\underline{x}^{(i)}$, $\hat{x}^{(i)}$.
2. For the initial values, we set $\underline{x}^{(1)} = 0$ and $\bar{x}^{(1)} = \hat{x}^{(1)} = x^*$, as the firm's optimal target leverage $\underline{x}$ is always positive and cannot exceed x^* . We then start with a linear equity value function: $p^{(1)}(x) = p^{(1)}(\underline{x}^{(1)}) + \frac{x - \underline{x}^{(1)}}{\hat{x}^{(1)} - \underline{x}^{(1)}} (p^{(1)}(\hat{x}^{(1)}) - p^{(1)}(\underline{x}^{(1)}))$ for $\underline{x}^{(1)} < x < \hat{x}^{(1)}$, so that $p^{(1)}(\underline{x}^{(1)}) = v^* - x^*$ and $p^{(1)}(\hat{x}^{(1)}) = 0$.
3. We then use the finite-difference method to solve the ODE for $p^{(i)}(x)$.
4. Change $p^{(i)}(\underline{x}^{(i)})$ and $p^{(i)}(\hat{x}^{(i)})$ to repeat step 3 until the boundary conditions (13) and (23) are satisfied.
5. Change $\underline{x}^{(i)}$ and $\hat{x}^{(i)}$ to repeat step 4 until the boundary conditions (14) and (24) are satisfied.
6. For a given $p^{(i)}(\hat{x}^{(i)})$, calculate $p^{(i)}(x)$ in the equity issuance region using (23).
7. Change $\bar{x}^{(i)}$ to repeat steps 3-6 until $p^{(i)}(\bar{x}^{(i)}) = 0$.

F Additional Results of Section 5

In this appendix, we provide additional results supporting our analyses in Section 5.

F.1 Details about the Effects of Equity Issuance Costs

To further investigate the channels underlying the comparative dynamics results in Table 3, Figure 9 plots the density of debt-EBIT ratio, $\frac{d\hat{F}(x)}{dx}$, and the market leverage function, $ML(x) = \frac{x}{x+p(x)}$, where $\hat{F}(x)$ is the pooled empirical distribution defined in (D.6). Panel A of Figure 9 shows that the density

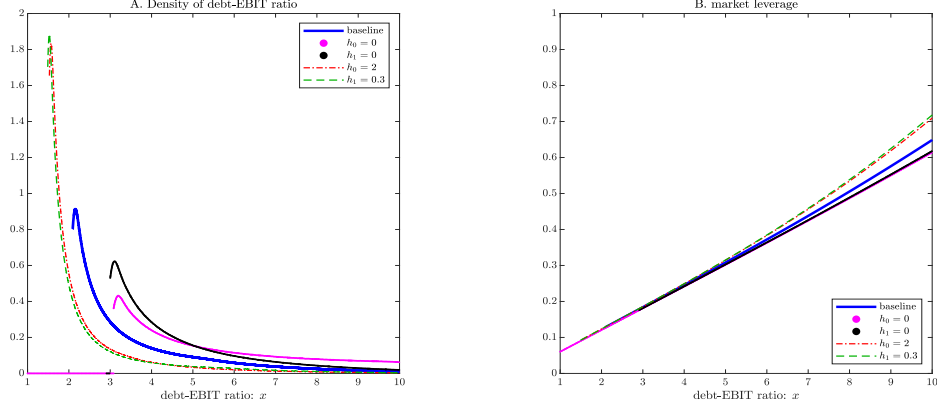


Figure 9: Effects of equity issuance costs on the density of the debt-EBIT ratio and market leverage.

Table 5: EFFECTS OF A JUMP'S (EXPECTED RISK-ADJUSTED) LOSS $\mathbb{E}(1 - Z)$ AND (RISK-ADJUSTED) ARRIVAL RATE λ ON MARKET LEVERAGE.

	Mean	Std. dev.	5%	50%	95%
baseline	0.23	0.12	0.13	0.19	0.50
20% (jump) loss	0.11	0.07	0.07	0.08	0.27
5% (jump) loss	0.30	0.15	0.16	0.25	0.62
$\lambda = 0.6$ (jump) rate	0.27	0.14	0.14	0.21	0.56
$\lambda = 0.3$ (jump) rate	0.30	0.15	0.16	0.24	0.63

of the debt-EBIT ratio shifts toward higher levels when the equity financing costs h_0 and h_1 decrease. Panel B of Figure 9 shows that market leverage, as a function of x , changes little as we vary h_0 and h_1 . Therefore, combining Table 3 with Figure 9, we conclude that the reported changes in market leverage in Table 3 are driven primarily by shifts in the density of the debt-EBIT ratio.

Combining Table 4 with Panel A of Figure 9, we see that the density of the debt-EBIT ratio is primarily determined by the target debt-EBIT ratio $\underline{x}$ and the recapitalization target $\tilde{x}$. When either $\underline{x}$ or $\tilde{x}$ is higher, the density of the debt-EBIT ratio becomes more concentrated at higher levels. For example, when $h_0 = 0$, both $\underline{x}$ and $\tilde{x}$ are the highest among the cases reported in Table 4, and the corresponding density of the debt-EBIT ratio is more concentrated at higher levels.

F.2 Effects of Jumps, Taxes, and Distress Costs on Leverage in Section 5.3

Effects of jumps on leverage. Two critical parameters drive jump risk in our model, λ the arrival rate of earnings shocks and β affecting the distribution of earnings losses Z under the risk-neutral measure (Recall that the distribution of recovery Z is the same under $\mathbb{P}$ and $\mathbb{Q}$.) For our comparative static analysis, we fix $\kappa = \ln(\lambda/\lambda^{\mathbb{P}})$ throughout. Therefore, decreasing the arrival rate λ under the risk-neutral measure $\mathbb{Q}$ by a certain fraction is the same as decreasing $\lambda^{\mathbb{P}}$ under the physical measure $\mathbb{P}$ by the same fraction.

Table 6: EFFECTS OF A JUMP’S (EXPECTED RISK-ADJUSTED) LOSS $\mathbb{E}(1 - Z)$ AND (RISK-ADJUSTED) ARRIVAL RATE λ ON THRESHOLDS.

	$\underline{x}$	$\tilde{x}$	x^0	$\check{x}$	$\hat{x}$	$\bar{x}$
baseline	2.1	5.3	8.9	10.4	12.6	14.7
20% (jump) loss	1.2	3.5	9.1	7.8	10.0	14.3
5% (jump) loss	2.5	6.5	5.6	12.4	14.1	14.9
$\lambda = 0.6$ (jump) rate	2.3	5.8	5.6	11.1	13.5	14.8
$\lambda = 0.3$ (jump) rate	2.5	6.3	4.6	11.9	14.5	14.9

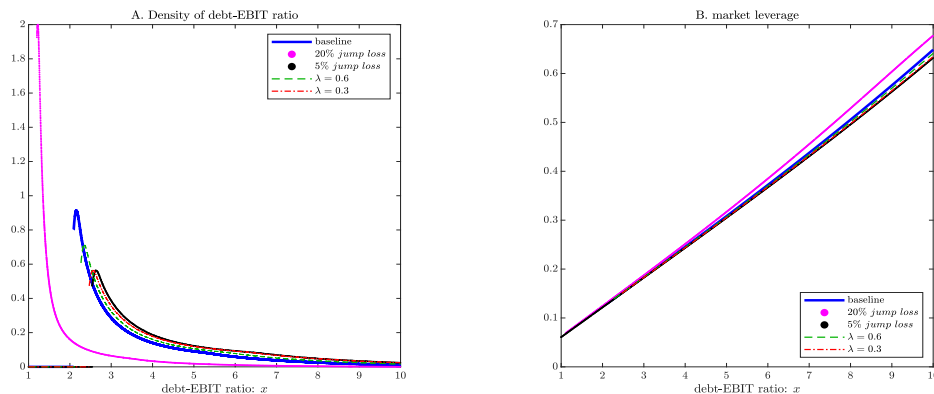


Figure 10: Effects of jump risk on the density of the debt-EBIT ratio and market leverage.

Table 5 shows that reducing the expected loss following the arrival of a jump $\mathbb{E}(1 - Z) = 1/(\beta + 1)$ by half from 10% to 5% increases average market leverage from 23% to 30%. A reduction in the arrival rate λ from 1.2 (once every 10 months) to 0.6 (once every 20 months) has a somewhat smaller effect. As Table 5 reports, average ML increases from 23% to just 27% as a result of this reduction in the arrival rate.

Panel A of Figure 10 shows that the density of the debt-EBIT ratio shifts toward lower levels as (expected) jump losses increase. By contrast, changes in the arrival rate λ have a relatively small effect on the density of the debt-EBIT ratio. Panel B of Figure 10 shows that the market leverage functions are broadly similar across different values of jump losses and the arrival rate λ . Therefore, combining Table 5 with Figure 10, we conclude that the reported changes in market leverage in Table 5 for different levels of jump loss are primarily driven by shifts in the density of the debt-EBIT ratio.

Table 6 shows that all trigger values— $\underline{x}$, $\tilde{x}$, $\check{x}$, $\hat{x}$, and $\bar{x}$ —are lower when jump losses are higher. This pattern is consistent with Panel A of Figure 10, which shows that the density of the debt-EBIT ratio becomes more concentrated at lower levels as jump losses increase. By contrast, changes in the arrival rate λ have only a limited effect on the target debt-EBIT ratio $\underline{x}$ and the recapitalization target $\tilde{x}$.

Effects of tax rates and default recovery on leverage. In Table 7, we explore the effect of changing the tax rate on leverage. For example, when the tax rate τ is increased from 21% to 30% median market leverage increases from 19% to only 22%. When the tax rate τ is lowered from 21% to

Table 7: EFFECTS OF TAXES (τ) AND LIQUIDATION RECOVERY φ (AS A FRACTION OF THE FIRM'S ENTERPRISE VALUE UNDER FIRST-BEST) ON MARKET LEVERAGE.

	Mean	Std. dev.	5%	50%	95%
baseline	0.23	0.12	0.13	0.19	0.50
$\tau = 10\%$	0.17	0.10	0.10	0.12	0.39
$\tau = 30\%$	0.27	0.13	0.15	0.22	0.54
$\varphi = 10\%$	0.22	0.12	0.13	0.18	0.48
$\varphi = 50\%$	0.25	0.13	0.14	0.20	0.54

Table 8: EFFECTS OF TAXES (τ) AND LIQUIDATION RECOVERY φ (AS A FRACTION OF THE FIRM'S ENTERPRISE VALUE UNDER FIRST-BEST) ON THRESHOLDS.

	$\underline{x}$	$\tilde{x}$	x^0	$\check{x}$	$\hat{x}$	$\bar{x}$
baseline	2.1	5.3	8.9	10.4	12.6	14.7
$\tau = 10\%$	1.7	5.4	9.5	11.0	13.6	16.5
$\tau = 30\%$	2.2	5.1	8.3	9.8	11.6	13.3
$\varphi = 10\%$	2.0	5.1	8.8	10.1	12.2	14.7
$\varphi = 50\%$	2.2	5.7	9.1	11.1	13.5	14.8

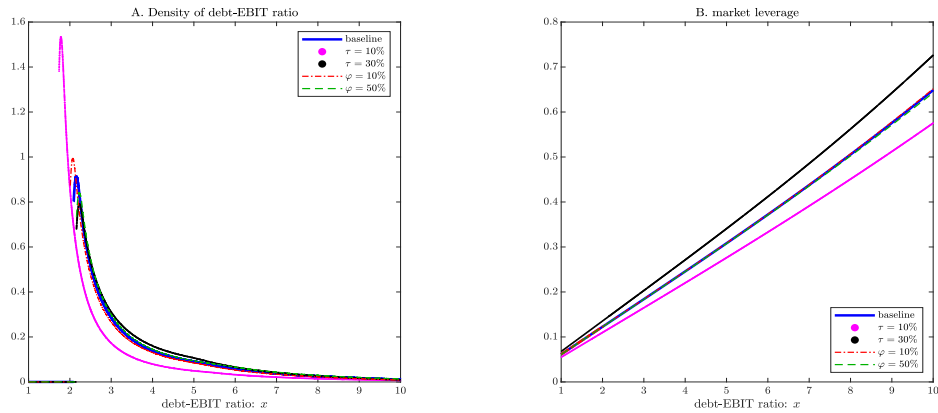


Figure 11: Effects of taxes and liquidation recovery on the density of the debt-EBIT ratio and market leverage.

10% median market leverage declines from 19% to 12%. Overall, we find some but limited effects of taxes on leverage.

Changes in the default recovery (as a fraction of the enterprise value under first-best) φ have an even smaller effect on leverage as we see from Table 7. When we increase the recovery fraction φ from

10% to 50%, median market leverage only increases from 18% to 20%.³⁹ The reason why corporate leverage policies are not very sensitive to changes of the default recovery value is that default is a low (even risk-adjusted) probability event, because the firm has an option to issue equity to deleverage.

As shown in Figure 11, a lower tax rate τ shifts the density of the debt-EBIT ratio toward lower levels and reduces the market leverage function, although the magnitude of this effect is modest. Changes in the default recovery fraction φ have an even smaller effect on the density of the debt-EBIT ratio and on the market leverage function. These results are consistent with Table 7. Indeed, Table 8 shows that all trigger values are insensitive to variation in the tax rate τ and the recovery fraction φ .

G Polar Cases with Respectively No Jumps and A Wipe-out Jump

In this appendix, we analyze two special cases on the opposite ends of jump effects: no jump at all ($\lambda = 0$) and 100% EBIT loss if a jump arrives ($Z = 0$).

G.1 No-jump ($\lambda = 0$)

If $\lambda = 0$, the EBIT $\{Y_t\}$ process is a diffusion. Debt is then risk free and the equilibrium credit spread equals zero: $\eta(x) = 0$. The ODE for $p(x)$ in the $x \in (\underline{x}, \hat{x})$ region then simplifies to:

$$(r - \mu)p(x) = [\theta(rx) - 1 + (r - \mu)x]p'(x) + \frac{\sigma^2 x^2}{2}p''(x). \quad (\text{G.1})$$

Since external equity is costly, the firm delays its equity issuance as much as possible. As a result, the equity-issuance boundary $\hat{x}$ equals the default boundary: $\hat{x} = \bar{x}$ so that $p(\hat{x}) = 0$. When tapping external equity, the firm issues the net amount m (per unit of EBIT) so that its leverage x reaches the recapitalization target $\tilde{x}$ where $p'(\tilde{x}) = -(1 + h_1)$. Equity value in the $x \leq \underline{x}$ region is given by (12) where the payout boundary $\underline{x}$ is characterized by (13) and (14) as in Section 4.⁴⁰

In Figure 12, we see that the firm is endogenously risk averse so that $v(x)$ is concave due to costly equity issuance. At $t = 0$, the firm (with no debt) increases its debt-EBIT ratio to $x_0 = \underline{x} = 2.88$ where market leverage is 17% through a lumpy dividend recapitalization with debt issuance of 2.88 per unit of EBIT. When x reaches the equilibrium debt limit: $\bar{x} = 15.04$, the firm's old shareholders are completely wiped out: $p(\bar{x}) = 0$ and the firm issues new equity by raising the net amount $m = 7.6$ while incurring an issuance cost of $h_0 + h_1 m = 1.3$ per unit of EBIT so that the $\{x_t\}$ process continues from $\tilde{x} = 7.44$, where $p'(\tilde{x}) = -(1 + h_1) = -1.09$, $v(\tilde{x}) = 16.3$, and market leverage is 45.5%.

G.2 Wipe-out jump: $Z = 0$

If a jump causes a 100% earnings loss, the firm defaults when the first jump arrives. The equilibrium credit spread then equals the (risk-neutral) jump arrival rate: $\eta(x) = \lambda$, so that $c(x) = (r + \lambda)x$. The firm's equity value, $p(x)$, satisfies the ODE in the $x \in (\underline{x}, \hat{x})$ region:

$$(r + \lambda - \mu)p(x) = [\theta((r + \lambda)x) - 1 + (r + \lambda - \mu)x]p'(x) + \frac{\sigma^2 x^2}{2}p''(x). \quad (\text{G.2})$$

³⁹ Note that $\varphi = 10\%$ corresponds to $\ell = 2$ and $\varphi = 50\%$ corresponds to $\ell = 10$ in terms of the recovery EBIT multiple ℓ . See footnote 8 in Section 2 for details of this mapping.

⁴⁰ Here, we focus on the case where the equity issuance cost is moderate. When equity issuance is too costly, the firm issues no equity. Then, the debt limit equals the firm's liquidation value: $\bar{x} = \ell$, debt is fully collateralized and hence is risk free.

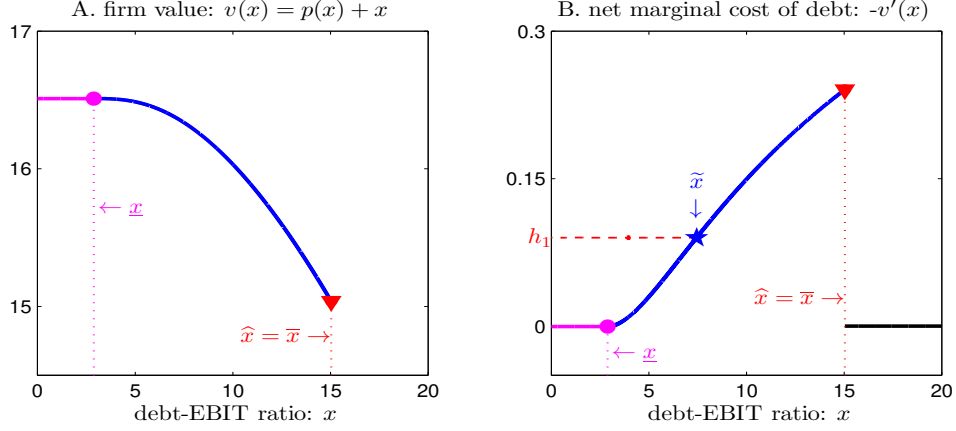


Figure 12: No-jump ($\lambda = 0$). Panels A and B plot firm value, $v(x)$, and the net marginal cost of debt, $-v'(x)$, respectively, for the $\lambda = 0$ case. Equity value $p(x)$ is concave because default does not happen.

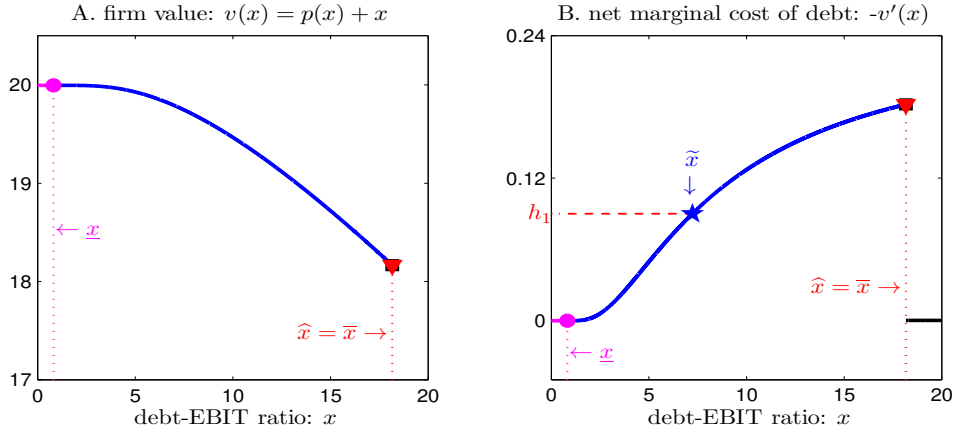


Figure 13: Wipe-out jump ($Z = 0$). Panels A and B plot firm value, $v(x)$, and the net marginal cost of debt, $-v'(x)$, respectively, for the $Z = 0$ case. Equity value $p(x)$ is concave because default is beyond the firm's control.

As in the no-jump case, the firm delays its equity issuance as much as possible by setting $\hat{x} = \bar{x}$ so that $p(\hat{x}) = 0$ and $p'(\hat{x}) = -(1 + h_1)$. Other conditions for the payout boundary $\underline{x}$ are the same as in the $\lambda = 0$ case.

In Figure 13, again we see that the firm is endogenously risk averse so that $v(x)$ is concave due to costly equity issuance. At $t = 0$, the firm starts from $x_0 = \underline{x} = 0.82$ where market leverage is 4.1% by financing a lumpy dividend payment with debt issuance of 0.82 per unit of EBIT. When x reaches the equilibrium debt limit: $\bar{x} = 18.17$, the firm's old shareholders are completely wiped out: $p(\bar{x}) = 0$. The firm issues new equity when x_t reaches the optimal recapitalization threshold $\hat{x}$, which is also the equilibrium debt limit: $\hat{x} = \bar{x} = 18.17$, by injecting a net amount $m = 10.94$ while incurring an issuance cost of $h_0 + h_1 m = 1.6$ per unit of EBIT into the firm so that the $\{x_t\}$ process continues from $\tilde{x} = 7.23$, where $p'(\tilde{x}) = -(1 + h_1) = -1.09$, $v(\tilde{x}) = 19.77$, and market leverage is 36.6%.

In sum, equity value $p(x)$ and hence firm value $v(x) = p(x) + x$ are globally concave and the firm is endogenously risk averse for all levels of x in both $\lambda = 0$ and $Z = 0$ cases. This is because in one case it is suboptimal to default when $\lambda = 0$ and in the other case the first jump arrival triggers default with probability one (that is, default is beyond the firm's control) when $Z = 0$. A key takeaway from these

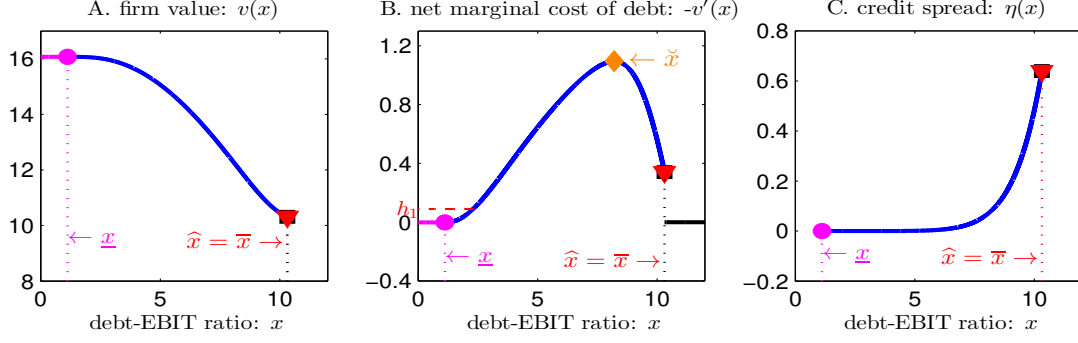


Figure 14: Fixed ($h_0 > 0$) and/or proportional (h_1) external equity issuance is sufficiently (but not too) costly. The firm issues equity only when $x = \hat{x} = \bar{x}$ and the equilibrium debt capacity is larger than the firm's liquidation value: $\bar{x} > \ell$. When $h_0 > 0$, equity value $p(x)$ is convex when x is high.

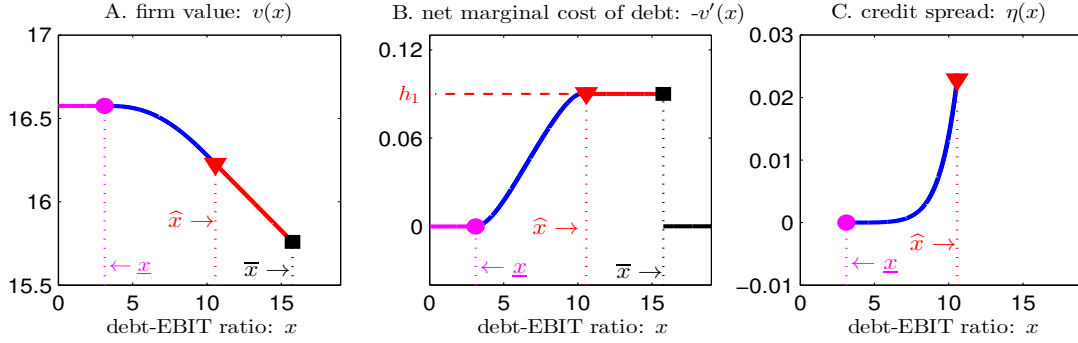


Figure 15: No fixed equity issuance cost ($h_0 = 0$) case. The optimal proactive deleveraging threshold and the recapitalization target are equal: $\hat{x} = \tilde{x} = 10.4$. Equity value $p(x)$ is concave.

two polar cases is that in order for a firm to have a risk seeking motive so that firm value may be locally convex in x , it is necessary that Y_t features *jumps* that cause a *fractional loss*: $\lambda > 0$ and $Z \in (0, 1)$, so that default is state-contingent.

H When Equity Issuance Costs are High or Zero

In this appendix, we analyze special cases where fixed and proportional equity issuance costs are alternately high or zero, and show that each of these special cases has key limitations in terms of the empirical plausibility of their predictions. We thus conclude that both fixed and proportional equity issuance costs play important roles in generating plausible leverage predictions.

H.1 High Fixed (h_0) and/or Proportional (h_1) Issuance Costs

When fixed ($h_0 > 0$) and/or proportional (h_1) equity issuance is sufficiently (but not too) costly, the firm still issues equity but only when $x = \hat{x} = \bar{x}$. That is, the equity-issuance region turns into a singleton. Since the firm can issue equity to continue its operations, its equilibrium debt capacity is larger than the firm's liquidation value: $\bar{x} > \ell$. But old shareholders are completely wiped out after equity issuance. Figure 14 illustrates these results with $h_0 = 5$.

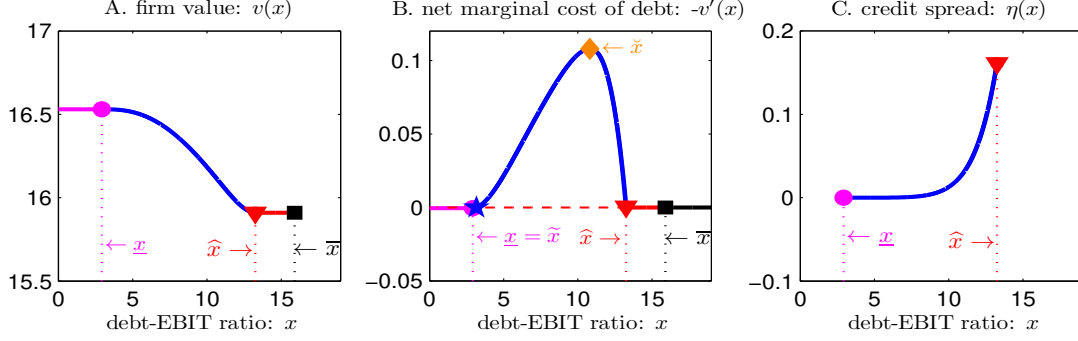


Figure 16: No proportional equity issuance cost ($h_1 = 0$) case. The optimal target and the recapitalization target are equal: $\underline{x} = \tilde{x} = 3.0$. Equity value $p(x)$ is convex when x is high.

H.2 No Fixed Equity Issuance Cost ($h_0 = 0$) and Moderate Proportional Cost h_1

Absent fixed equity issuance costs, equity value $p(x)$ is concave in x as the firm has no risk seeking incentive. Also, the equity issuance boundary $\hat{x}$ is equal to the recapitalization target $\tilde{x}$. Figure 15 plots the solution for the case where $h_1 = 9\%$ and confirms these results.

At $t = 0$, the firm starts with $x_0 = \underline{x} = 3.11$ and an implied market leverage of 18.8% by financing a dividend payment with debt. When x hits $\bar{x} = 15.6$, the firm exhausts its debt capacity and old shareholders are completely wiped out. When a jump results in x entering the region $(\hat{x}, \bar{x}) = (10.4, 15.6)$, the firm issues an amount of external equity to bring leverage x down to $\hat{x} = 10.4$. Since $h_1 = 9\%$, the net marginal cost of debt at all x lies inside the $[0, 9\%]$ region. The higher the level of x , the higher the marginal cost of debt: $-v'(x)$.

H.3 No Proportional Equity Issuance Cost ($h_1 = 0$) and Moderate Fixed Cost h_0

What happens if $h_1 = 0$ and h_0 is strictly positive but not high? In this case, firm value is not concave due to the fixed equity issuance cost. The recapitalization target ($\hat{x}$) and the optimal target ($\underline{x}$) are equal. Figure 16 confirms these results. However, this prediction is implausible as equity is more subject to information sensitivity (Myers and Majluf, 1984).

I A Model with Dynamic Hedging and Risk Taking

What if the firm can trade a risky asset (e.g., futures) to manage its risk exposure? In this appendix, we extend our baseline model of Section 2 by introducing a risky financial asset (futures). Absent default, over dt , the firm's debt balance is stochastic and evolves as:

$$dX_t = (C_{t-} + \Theta_{t-} - Y_{t-}) dt - dN_t + dU_t + \Omega_{t-} dZ_t, \quad (\text{I.1})$$

where the last term captures the gains (losses) from trading futures under competitive markets (Duffie, 2001). Let ρ denote the correlation between the shock Z_t and B_t , the standard Brownian motion that drives the EBIT process (1). Without loss of generality, let $\rho \geq 0$.

Under frictionless futures markets, the firm's profits (losses) from its futures position are given by $\Omega_{t-} dZ_t$, where Ω_{t-} is the futures position and Z_t is a standard Brownian motion under the risk-neutral measure (Duffie, 2001). Let $\omega_{t-} = \Omega_{t-}/Y_{t-}$. We assume that the constraint on the firm's futures risk

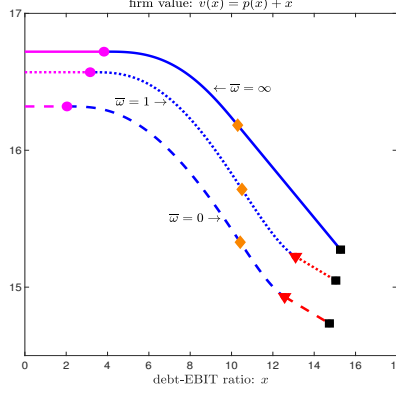


Figure 17: Firm value $v(x)$ in a model with hedging/risk taking opportunities.

exposure is given by $|\Omega_{t-}| \leq \bar{\omega} Y_{t-}$ or equivalently $|\omega_{t-}| \leq \bar{\omega}$ for all $t \geq 0$, where $\bar{\omega} \geq 0$ measures the tightness of this constraint.

In the earnings-retention region, the firm chooses ω to solve the following HJB equation:

$$\begin{aligned} rP(Y, X) = \max_{\omega} \quad & \mu Y P_Y(Y, X) + (C + \Theta - Y) P_X(Y, X) + \frac{\sigma^2 Y^2}{2} P_{YY}(Y, X) \\ & + \frac{\omega^2 Y^2}{2} P_{XX}(Y, X) + \rho \sigma \omega Y^2 P_{XY}(Y, X) + \lambda \mathbb{E} [P(ZY, X) - P(Y, X)] , \end{aligned} \quad (\text{I.2})$$

subject to the constraint $|\omega| \leq \bar{\omega}$. Using the homogeneity property to simplify (I.2), we obtain:

$$\begin{aligned} rp(x) = \max_{|\omega| \leq \bar{\omega}} \quad & [c(x) + \theta(c(x)) - 1] p'(x) + \mu(p(x) - xp'(x)) + \frac{(\sigma^2 x^2 - 2\rho\omega\sigma x + \omega^2)}{2} p''(x) \\ & + \lambda \mathbb{E} [Zp(x/Z) - p(x)] . \end{aligned} \quad (\text{I.3})$$

Note that choosing ω in (I.3) boils down to solving the following constrained (static) problem:

$$\max_{|\omega| \leq \bar{\omega}} \quad (\sigma^2 x^2 - 2\rho\omega\sigma x + \omega^2) p''(x) . \quad (\text{I.4})$$

When the firm is endogenously risk averse, so that $p''(x) < 0$, the optimal policy is $\omega = \min\{\rho\sigma x, \bar{\omega}\}$ as $\rho \geq 0$. That is, absent hedging costs and when the constraint is slack ($\rho\sigma x \leq \bar{\omega}$), it is optimal to eliminate the firm's hedgeable risk exposure and retain only unhedgeable (idiosyncratic and jump) risk. However, when the firm is endogenously risk seeking, so that $p''(x) > 0$, the firm seeks maximal risk so that the optimal policy is $\omega = -\bar{\omega}$ (as $\rho \geq 0$). Finally, we verify that the optimal policy ω is consistent with the sign of $p''(x)$.

In Figure 17, we show that as we relax the constraint $|\omega_{t-}| \leq \bar{\omega}$, the firm can manage its risk exposure more effectively, issues more debt, and pays dividends sooner, so that the degree of firm value convexity (in the high-leverage region) decreases.⁴¹ In the limit, as its risk exposure (ω_{t-}) becomes unconstrained ($\bar{\omega} \rightarrow \infty$), the firm randomizes over x so much that $v(x)$ becomes (weakly) globally concave. In sum, using the risky asset to randomize is a more efficient way to manage leverage dynamics and allows the firm to do away with costly equity issuance in the limit.

⁴¹ We set $\rho = 0.8$, and use the same values for all other parameters as in subsection 4.5.

J Growth Options, Capital Accumulation, and Leverage

Corporate investments feature both small incremental adjustments to capital stock (the intensive margin) and large lumpy ‘growth option’ increments (the extensive margin).⁴² In this appendix, we model both types of investment decisions by extending our EBIT model of Section 2 to incorporate (1.) a stochastic growth option arrival and (2.) convex capital adjustment costs. This extended model reveals non-monotonic and nonlinear effects of leverage on the exercise of growth options and capital accumulation, generating new predictions on the joint dynamics of investment, leverage, equity issuance, asset sales, and payouts.

J.1 Model with Capital Accumulation

As in our baseline model, we directly work under the risk-neutral measure. A firm starts off at $t = 0$ with an asset in place (capital). First, to model the intensive margin of day-to-day capital adjustments, we use a q theory of investment. Let K and I denote the capital stock and gross investment, respectively.

Second, we introduce the intensive margin of corporate investment. A growth option arrives at stochastic time T^G at a constant rate of $\lambda_G > 0$ per annum. Upon its arrival, the firm can either exercise the option to permanently enhance its productivity or bypass this option. Exercising the option enhances the firm’s productivity but involves a one-time upfront cost: ϕK_{T^G} , where $\phi > 0$. By making the investment cost proportional to K_{T^G} , we ensure that the value of this growth option does not grow out of this upgrade cost, so that the benefit and cost of exercising the growth option are comparable.⁴³

At any time t , the firm is in one of the three phases: $s_t \in \{0, 1, 2\}$. The firm’s unlevered free cash flow net of investment in phase s_t is given by:

$$Y_t = A_{s_t} K_t - I_t, \quad (\text{J.1})$$

where A_{s_t} denotes the firm’s productivity in phase s_t . Before the growth option arrives ($t < T^G$), it is in the growth phase: $s_t = 0$. After the arrival ($t \geq T^G$), the firm is in one of the two mature phases. If the growth option is exercised at T^G , it is in the $s_t = 2$ phase with productivity $A_2 > A_0$. If the growth option is bypassed at T^G , the firm is in the $s_t = 1$ phase and its productivity is unchanged: $A_1 = A_0$.

In all three phases the capital stock evolves according to:

$$dK_t = \Psi(I_{t-}, K_{t-})dt + \sigma_K K_{t-} d\mathcal{B}_t^K - (1 - Z)K_{t-} d\mathcal{J}_t^K, \quad (\text{J.2})$$

where $\sigma_K > 0$ is the diffusion-volatility parameter, and $\mathcal{B}^K$ is a standard Brownian motion. These continuous shocks can be thought of as stochastic capital depreciation shocks as in Barro (2006). Following Lucas and Prescott (1971), Hayashi (1982), and Jermann (1998), we assume that $\Psi(I, K)$ is homogeneous of degree one in I and K :

$$\Psi(I, K) = \psi(i) \cdot K, \quad (\text{J.3})$$

where $i = I/K$, $\psi'(i) > 0$, and $\psi''(i) < 0$.

⁴² Models that analyze the effects of financial frictions on growth options include Philippon and Sannikov (2007), Hugonnier, Malamud, and Morellec (2015), Malenko (2019), and Bolton, Wang, and Yang (2019). Philippon and Sannikov (2007) analyze the optimal timing and financing of large, irreversible investments in the presence of moral hazard. Hugonnier, Malamud, and Morellec (2015) analyze the effect of capital supply uncertainty on lumpy growth option exercising. Malenko (2019) analyzes dynamic capital budgeting when the manager with empire-building preferences privately observes the arrival and properties of growth options. Bolton, Wang, and Yang (2019) incorporate costly external equity into a classic real-option setting of McDonald and Siegel (1986) and Dixit and Pindyck (1994).

⁴³ This is a common assumption in the investment literature, see, e.g., Cooper and Haltiwanger (2006). If the fixed investment cost is independent of firm size, it will not matter when firms become sufficiently large in the long run.

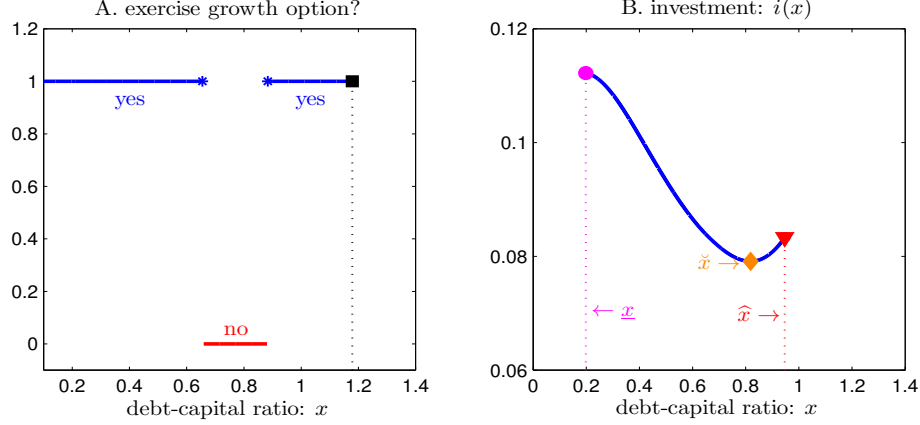


Figure 18: This figure shows that the firm's growth option exercising and investment policies are non-monotonic in book leverage x in the growth phase ($s_t = 0$).

The third term in (J.2) describes the risk with respect to discrete downward jumps in the level of capital stock K the firm is exposed to, where $\mathcal{J}^K$ is a jump process with a constant arrival rate $\lambda_K > 0$. If a jump occurs at time t , so that $d\mathcal{J}_t^K = 1$, the capital stock drops from K_{t-} to $K_t = ZK_{t-}$. As before, $F(Z)$ denotes a well-behaved cumulative distribution function for $Z \in [0, 1]$. At the moment of default T^D , creditors receive the firm's liquidation value $L_{T^D} = \ell_K K_{T^D}$, where $\ell_K > 0$ and shareholders are wiped out. With a low recovery value ℓ_K , default generates deadweight losses.

To preserve our generalized model's homogeneity property, we assume that when issuing equity the firm pays a fixed cost of $h_0^K K_t \geq 0$ and a constant marginal cost $h_1 \geq 0$ so that the total equity issuance cost is $h_0^K K_t + h_1 M_t$ when the firm raises a net amount M_t . The tax schedule and characterization of the equilibrium credit spread are as in Section 2. In Internet Appendix ID, we solve the model using backward induction.⁴⁴ For brevity, we focus our discussion in this section on the growth ($s_t = 0$) phase.

J.2 Financing Growth Option and Investment: New Predictions

Anticipating that a valuable growth option may stochastically arrive and the prospect of a large expenditure that could entail a costly equity issuance, how should the firm manage its on-going investment and financing? And under what circumstances is the firm able to pay for the large expenditure from exercising the growth option and if so, in which form: debt or equity? Note that in this subsection we let $x_t = X_t/K_t$ denote book leverage as opposed to the debt-EBIT ratio.

In Figure 18, we plot the firm's growth option exercise decision and smooth capital investment policies, in a setting where financing policies critically matter. Both the growth option exercise and investment are *non-monotonic* in leverage.⁴⁵ When leverage is high or low, the firm exercises the growth option when it arrives, but bypasses the growth option when leverage is at an intermediate level. What is the intuition for these *non-monotonicity* results?

When leverage is low, the firm can finance the growth option with debt, because the (marginal) cost of debt $-v'(x)$ is low. When leverage is at an intermediate level, the shadow cost of debt $-v'(x)$ is high

⁴⁴ As the firm's decisions in the growth phase ($s_t = 0$) depend on its decisions in the two mature phases, firm value is higher in the former than in the latter, even though productivity is the same in the growth phase ($s_t = 0$) and the mature phase ($s_t = 1$) in which the firm did not exercise its growth option.

⁴⁵ This non-monotonicity result requires that the option exercising cost and the external equity financing cost are economically comparable. This is the case for Figure 18. Appendix ID.6 provides details for the parameter values.

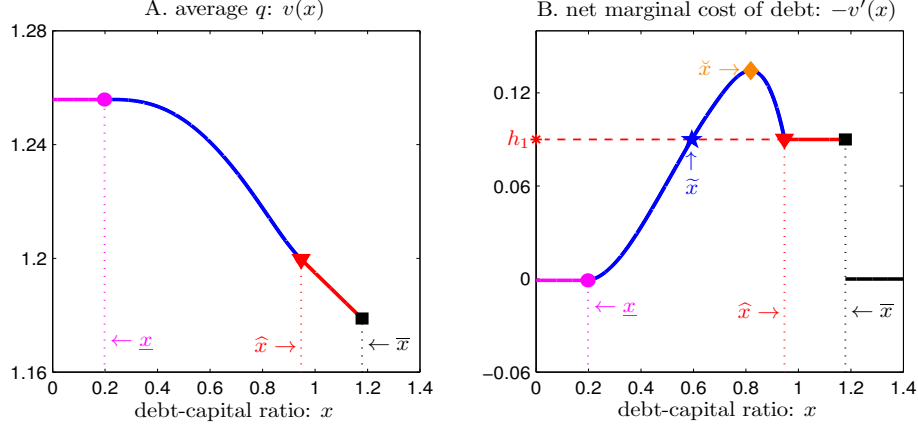


Figure 19: The firm's average q in the growth phase ($s_t = 0$), $v(x)$, is concave in the $x \in (\underline{x}, \check{x})$ sub-region but convex in the $x \in (\check{x}, \hat{x})$ sub-region of the target zone. Panel B makes the sign of $v''(x)$ clear by showing that the marginal cost of debt, $-v'(x)$, is non-monotonic in x .

(see Panel B of Figure 19). The firm then bypasses the growth option because financing the growth option expenditure with debt increases leverage too much. The firm cuts its investment more as x increases in this region (see Panel B of Figure 18). Bypassing the growth option and underinvesting in capital reflect a significant debt overhang as indicated by a high (shadow) cost of debt: $-v'(x)$.

Finally, when leverage is high, the firm is in a debt spiral, a dire financial situation. The firm taps costly external equity because there are two benefits of issuing equity: financing the growth option expenditure and at the same time deleveraging the balance sheet, which makes the firm financially more resilient (see the third region in Panel A of Figure 18). Anticipating that its leverage will discretely adjust downward when the firm exercises the growth option, the firm optimally reduces its underinvestment as x increases in the high x region (see Panel B of Figure 18).

In sum, the exercise of growth options and investment have implications on both value creation and prudent debt (and risk) management. When exercising a growth option calls for a lumpy cost, external equity financing can become a key source of financing to avoid excessively high leverage. This is because the exercise of growth options and standard capital accumulation policies are affected by *debt overhang* (Myers, 1977) and *risk shifting* (Jensen and Meckling, 1976) effects, the latter being driven by the interaction between downward jumps and fixed equity issuance costs.

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Internet Appendices to “Leverage and Credit Risk Dynamics under Costly Equity Issuance”

IA Proofs of Main Results

Proof of Lemma 1. Fix an MPE S^* whose default region is

$$\mathcal{D} = \{(Y, X) : X/Y \in \mathbb{X}^{\mathcal{D}}\},$$

where $\mathbb{X}^{\mathcal{D}} \subseteq [\ell, \infty)$ is open. Let $P(Y, X) = \mathcal{P}(Y, X; S^*, S^*)$ denote the corresponding equity value.

Fix an arbitrary $Y > 0$ throughout the proof. Limited liability implies $P(Y, X) \geq 0$. Moreover, if $X/Y \in \mathbb{X}^{\mathcal{D}}$, then $X \geq \ell Y$ and therefore $P(Y, X) = 0$.

Step 1: We prove that $(x, \infty) \subseteq \mathbb{X}^{\mathcal{D}}$ for some $x > 0$.

Choose $x_0 \notin \mathbb{X}^{\mathcal{D}}$ and set $X_0 = Yx_0$. For any $x_1 > x_0$ with $x_1 \notin \mathbb{X}^{\mathcal{D}}$, set $X_1 = Yx_1$. Starting from (Y, X_0) , the firm can issue additional debt and make a lump-sum payout $X_1 - X_0$, moving directly to (Y, X_1) . Because both states lie outside the default region, this is a feasible deviation. Equilibrium optimality gives

$$P(Y, X_0) \geq P(Y, X_1) + X_1 - X_0 \geq X_1 - X_0.$$

Consequently,

$$x_1 \leq x_0 + \frac{P(Y, Yx_0)}{Y}.$$

Since $P(Y, Yx_0)$ is finite, the set $(x_0, \infty) \setminus \mathbb{X}^{\mathcal{D}}$ is bounded above. Hence there exists $x < \infty$ such that $(x, \infty) \subseteq \mathbb{X}^{\mathcal{D}}$.

Step 2: The default region is connected.

Define

$$a = \inf \mathbb{X}^{\mathcal{D}}, \quad b = \inf \{x : (x, \infty) \subseteq \mathbb{X}^{\mathcal{D}}\}.$$

Step 1 implies that $b < \infty$ and $(b, \infty) \subseteq \mathbb{X}^{\mathcal{D}}$. Since $\mathbb{X}^{\mathcal{D}}$ is open, both a and b lie outside $\mathbb{X}^{\mathcal{D}}$.

Suppose, toward a contradiction, that $a < b$. Write $A = Ya$ and $B = Yb$. Starting from (Y, A) , the firm can issue debt and pay out $B - A$, moving directly to (Y, B) . Therefore,

$$P(Y, A) \geq P(Y, B) + B - A \geq B - A > 0. \tag{IA.1}$$

We first consider the case $h_0 = 0$. Choose $\delta \in (0, \frac{P(Y, A)}{1+h_1})$. Starting from any state (Y, X) with $A < X < A + \delta$, the firm can issue equity with net proceeds $X - A$ and reduce its debt to A . Since (Y, A) lies outside the conjectured default region, this deviation is feasible and yields

$$P(Y, A) - (1 + h_1)(X - A) > 0.$$

Equilibrium optimality therefore implies

$$P(Y, X) > 0, \quad A < X < A + \delta.$$

Next, consider the case $h_0 > 0$. Because $b > a \geq \ell$, we have $B > \ell Y$. At (Y, B) , an increase in debt would place the firm in the upper-tail default region. Furthermore, without an immediate equity intervention, the nondegenerate diffusion component of X_t/Y_t would enter (b, ∞) immediately. Thus, the admissibility requirement at a nondefault boundary requires an immediate equity intervention. Since $h_0 > 0$, admissible equity issuance is lumpy. Hence there exists $\tilde{B} < B$ such that

$$P(Y, B) = P(Y, \tilde{B}) - h_0 Y - (1 + h_1)(B - \tilde{B}) \geq 0. \quad (\text{IA.2})$$

In particular, $P(Y, \tilde{B}) > 0$, so $\tilde{B}/Y \notin \mathbb{X}^{\mathcal{D}}$.

If $\tilde{B} \leq A$, then starting from any $X \in (A, B)$ the firm can issue equity and reduce its debt to $\tilde{B}$. Therefore,

$$\begin{aligned} P(Y, X) &\geq P(Y, \tilde{B}) - h_0 Y - (1 + h_1)(X - \tilde{B}) \\ &= P(Y, B) + (1 + h_1)(B - X) \\ &> 0, \end{aligned}$$

where the equality uses (IA.2).

If $\tilde{B} > A$, then the firm can move from (Y, A) to $(Y, \tilde{B})$ by issuing debt and paying out $\tilde{B} - A$. Thus,

$$P(Y, A) \geq P(Y, \tilde{B}) + \tilde{B} - A.$$

Starting instead from any (Y, X) with $X > A$, the firm can issue equity and reduce its debt to A . Combining these two feasible deviations gives

$$\begin{aligned} P(Y, X) &\geq P(Y, A) - h_0 Y - (1 + h_1)(X - A) \\ &\geq P(Y, \tilde{B}) + \tilde{B} - A - h_0 Y - (1 + h_1)(X - A) \\ &= P(Y, B) + (1 + h_1)(B - \tilde{B}) + \tilde{B} - A - (1 + h_1)(X - A) \\ &\geq B - A - (1 + h_1)(X - A). \end{aligned}$$

The last expression is strictly positive whenever $0 < X - A < \frac{B-A}{1+h_1}$.

Thus, in all cases, there exists $\delta > 0$ such that

$$P(Y, X) > 0, \quad A < X < A + \delta.$$

However, since $a = \inf \mathbb{X}^{\mathcal{D}}$, there exists a sequence $x_n \in \mathbb{X}^{\mathcal{D}}$ with $x_n > a$ and $x_n \rightarrow a$. Setting $X_n = Yx_n$, we have $X_n \rightarrow A$ from above, while

$$P(Y, X_n) = 0$$

because $x_n \in \mathbb{X}^D$. This contradicts the strict positivity established above. Therefore, $a = b$.

Combining Steps 1 and 2 and using the openness of $\mathbb{X}^D$, we obtain $\mathbb{X}^D = (b, \infty)$. Setting $\bar{x} = b \geq \ell$ proves the lemma. $\blacksquare$

In the following proofs, we use a pair $(S, \tilde{S})$ to represent the firm's strategy S and creditors' belief $\tilde{S}$ such that $S \in \Phi(\tilde{S})$.

Proof of Proposition 1.

Step 1: We prove that (B.2) holds.

Starting from any state (Y, X) , consider a Markov strategy $S = (T^D, \{N_t, U_t; t \in (0, T^D)\})$ such that $S \in \Phi(\tilde{S})$. Let $(Y_t^S, X_t^S)_{t \geq 0}$ denote the state processes under the strategy pair $(S, \tilde{S})$. Fix any constant $k > 0$. Starting from state (kY, kX) , we denote strategy $kS = (T^{D^k}, \{N_t^k, U_t^k; t \in (0, T^{D^k})\})$, where $U_t^k - U_0^k = k(U_t - U_0)$, $N_t^k - N_0^k = k(N_t - N_0)$ and $T^{D^k} = \inf\{t \geq 0 : (Y_t^S, X_t^S) \in \mathcal{D} \mid (Y_0^S, X_0^S) = (Y, X)\}$. Since $\tilde{S} \in \mathcal{S}_x$, its default region is given by $\tilde{D} = \{(Y, X) : X/Y \in \mathbb{X}^{\tilde{D}}\}$, where $\mathbb{X}^{\tilde{D}} = (\bar{x}, \infty)$ for some $\bar{x} \geq \ell$. We derive from (A.6) that credit spread at time t with state (Y, X) is given by $\eta(x) = \lambda \mathbb{E}[(1 - \frac{\ell Z}{x})^+ \mathbf{1}_{x/Z \in \mathbb{X}^{\tilde{D}}}]$, where $x = X/Y$. Combining with the homogeneity property of Y , we conclude that the state processes (Y_t^{kS}, X_t^{kS}) under the strategy pair $(kS, \tilde{S})$ satisfy $(Y_t^{kS}, X_t^{kS}) = (kY_t^S, kX_t^S)$ when $(Y_0^{kS}, X_0^{kS}) = (kY, kX)$ and $(Y_0^S, X_0^S) = (Y, X)$. It follows that $\mathcal{P}(kY, kX; kS, \tilde{S}) = k\mathcal{P}(Y, X; S, \tilde{S})$.

In particular, for $S^* = \arg \max_{\hat{S}} \mathcal{P}(Y, X; \hat{S}, \tilde{S})$, we have

$$\mathcal{P}(kY, kX; kS^*, \tilde{S}) = k\mathcal{P}(Y, X; S^*, \tilde{S}) \geq k\mathcal{P}(Y, X; 1/k\hat{S}, \tilde{S}) = \mathcal{P}(kY, kX; \hat{S}, \tilde{S}), \quad (\text{IA.3})$$

for any $\hat{S} \in \Phi(\tilde{S})$. It follows that $k\mathcal{P}(Y, X; S^*, \tilde{S}) = \max_{\hat{S}} \mathcal{P}(kY, kX; \hat{S}, \tilde{S})$. That is, $k\tilde{P}(Y, X) = \tilde{P}(kY, kX)$. Let $k = 1/Y$, so that we obtain (B.2).

In the following, we denote $\mathbb{X}^D$ as the scaled default region under strategy S^* .

Step 2: We prove that $\mathbb{X}^D = (v, \infty)$, and $\tilde{p}(x) = 0$ for any $x \geq v$, where $v = \inf\{x > 0 : \tilde{p}(x) = 0\}$ and $\tilde{p}(x) = \tilde{P}(1, x)$.

Under shareholder limited liability, we have $\tilde{p}(x) \geq 0$ for any $x > 0$. For the sake of contradiction, assume there exists $x_1 < v$ such that $x_1 \in \mathbb{X}^D$. Since $x \geq \ell$ for any $x \in \mathbb{X}^D$, we have $\tilde{p}(x_1) = \max\{\ell - x_1, 0\} = 0$. Then we arrive at a contradiction as $v = \inf\{x > 0 : \tilde{p}(x) = 0\}$. Thus, $\mathbb{X}^D \subseteq [v, \infty)$.

For the sake of contradiction, assume $\tilde{p}(x) > 0$ for some $x > v$. Let $x_0 = \inf\{x \geq v : \tilde{p}(x) > 0\}$. Then for any $\varepsilon > 0$, there exists $x_1 < x_2$, such that $|x_i - x_0| \leq \varepsilon$, $i = 1, 2$ and $\tilde{p}(x_1) = 0$, $\tilde{p}(x_2) > 0$.

Recall $\mathbb{X}^{\tilde{D}} = (\bar{x}, \infty)$, where $\bar{x} \geq \ell$. If $x_1 \geq \bar{x}$, then $x_2 \in (\bar{x}, \infty) = \mathbb{X}^{\tilde{D}}$. Since $\tilde{p}(x_2) > 0 = \max\{\ell - x_2, 0\}$, firm does not default at x_2 . According to Definition 2, the firm must issue equity to immediately exit $\tilde{D}$. Thus, there exists $\tilde{x}_2 \leq \bar{x}$, such that $\tilde{p}(x_2) = \tilde{p}(\tilde{x}_2) - (1 + h_1)(x_2 - \tilde{x}_2) - h_0$. Starting from $X_t/Y_t = x_1$, the firm can issue equity $x_1 - \tilde{x}_2$, and receive

$$\tilde{p}(\tilde{x}_2) - (1 + h_1)(x_1 - \tilde{x}_2) - h_0 > \tilde{p}(\tilde{x}_2) - (1 + h_1)(x_2 - \tilde{x}_2) - h_0 = \tilde{p}(x_2) > 0, \quad (\text{IA.4})$$

where the first inequality uses $x_1 < x_2$. Since $\tilde{p}(x)$ is the optimal equity value, the left side of (IA.4) is no larger than $\tilde{p}(x_1)$. It follows from (IA.4) that $\tilde{p}(x_1) > 0$. Then we arrive at a contradiction as $\tilde{p}(x_1) = 0$.

If $x_1 < \bar{x}$, then $x_2 < \bar{x}$ for sufficiently small ε . Since $[x_1, x_2] \cap \mathbb{X}^{\tilde{\mathcal{D}}} = \emptyset$, starting from $X_t/Y_t = x_1$, the firm can make a payout $x_2 - x_1$, and shareholders receive $x_2 - x_1 + \tilde{p}(x_2) > 0$. Since $\tilde{p}(x)$ is the optimal equity value, we have $\tilde{p}(x_1) \geq x_2 - x_1 + \tilde{p}(x_2) > 0$. Then we arrive at a contradiction as $\tilde{p}(x_1) = 0$.

Therefore, $\tilde{p}(x) = 0$ for any $x > v$ and $(v, \infty) \subseteq \mathbb{X}^{\mathcal{D}}$. In addition, it is clear that $\tilde{p}(v) = 0$. Since we have shown that $\mathbb{X}^{\mathcal{D}} \subseteq [v, \infty)$, we have $\mathbb{X}^{\mathcal{D}} = (v, \infty)$. ■

Proof of Proposition 2. In the following, we denote $S_{\bar{x}}$ as the strategy with default region $\{(Y, X) : X/Y > \bar{x}\}$, and denote $S_{\bar{x}}^*$ as the optimal strategy such that $\mathcal{P}(Y, X; S_{\bar{x}}^*, S_{\bar{x}}) \geq \mathcal{P}(Y, X; \hat{S}, S_{\bar{x}})$ for all $(Y, X) \in \Sigma$ and $\hat{S} \in \Phi(S_{\bar{x}})$.

Step 1: We show that for $\mathbb{X}^{\mathcal{D}_1} \supseteq \mathbb{X}^{\mathcal{D}_2}$, strategy $\tilde{S}_i$ with default region $\{(Y, X) : X/Y \in \mathbb{X}^{\mathcal{D}_i}\}$ and $S_i = \arg \max_S \mathcal{P}(Y, X; S, \tilde{S}_i)$, $i = 1, 2$, we have

$$\mathcal{P}(Y, X; S_1, \tilde{S}_1) \leq \mathcal{P}(Y, X; S_2, \tilde{S}_2). \quad (\text{IA.5})$$

Let $\eta_i(x) = \lambda \mathbb{E}[(1 - \frac{\ell Z}{x})^+ \mathbf{1}_{x/Z \in \mathbb{X}^{\mathcal{D}_i}}]$. Since $\mathbb{X}^{\mathcal{D}_1} \supseteq \mathbb{X}^{\mathcal{D}_2}$, we have $\eta_1(x) \geq \eta_2(x)$ for any $x \notin \mathbb{X}^{\mathcal{D}_1}$. Let the sum of (scaled) coupon and tax payments be given by

$$\Lambda(x, \eta) := (r + \eta)x + \tau(1 - (r + \eta)x)\mathbf{1}_{(r+\eta)x \leq 1}. \quad (\text{IA.6})$$

Since $\tau \in (0, 1)$, $\Lambda(x, \eta)$ is nondecreasing in the credit spread η .

Let $S_1 = (T^{\mathcal{D}_1}, \{N_t^1, U_t^1; t \in (0, T^{\mathcal{D}_1})\})$. Consider the strategy $\hat{S} = (T^{\mathcal{D}_1}, \{N_t^1, \hat{U}_t^1; t \in (0, T^{\mathcal{D}_1})\})$, where $\hat{U}_t^1$ is given by

$$\hat{U}_t^1 = U_t^1 + \int_0^t Y_s [\Lambda(x_s, \eta_1(x_s)) - \Lambda(x_s, \eta_2(x_s))] ds, \quad (\text{IA.7})$$

where $x_s = X_s/Y_s$ and (Y_s, X_s) are states derived from the firm's strategy S_1 and creditors' belief $\tilde{S}_1$. By definition (IA.7), one can see that $\{(Y_s, X_s)\}_{s \geq 0}$ are the same under strategy pair $(S_1, \tilde{S}_1)$ and strategy pair $(\hat{S}, \tilde{S}_2)$. Thus,

$$\begin{aligned} \mathcal{P}(Y, X; S_2, \tilde{S}_2) &= \max_S \mathcal{P}(Y, X; S, \tilde{S}_2) \geq \mathcal{P}(Y, X; \hat{S}, \tilde{S}_2) \\ &= \mathcal{P}(Y, X; S_1, \tilde{S}_1) + \mathbb{E} \left[\int_0^{T^{\mathcal{D}_1}} e^{-rs} Y_s [\Lambda(x_s, \eta_1(x_s)) - \Lambda(x_s, \eta_2(x_s))] ds \right] \\ &\geq \mathcal{P}(Y, X; S_1, \tilde{S}_1), \end{aligned} \quad (\text{IA.8})$$

where the second inequality uses $\eta_1(x) \geq \eta_2(x)$ and the property that $\Lambda(x, \eta)$ is nondecreasing in η .

Using (B.2), we note that for $x = X/Y$, and $p(x; \bar{x}) := \mathcal{P}(1, x; S_{\bar{x}}^*, S_{\bar{x}})$,

$$p(x; \bar{x}) = \mathcal{P}(Y, X; S_{\bar{x}}^*, S_{\bar{x}})/Y. \quad (\text{IA.9})$$

Define

$$\mathcal{A}(\bar{x}) = \inf\{x > 0 : p(x; \bar{x}) = 0\}. \quad (\text{IA.10})$$

Using the results proved in step 1, we can see that $\mathcal{A}(\bar{x})$ is nondecreasing in $\bar{x} \geq \ell$.

Step 2: We prove that $\mathcal{A}(\bar{x}) < \bar{x}$ for all $\bar{x} > \frac{1}{r-g}$.

Recall that strategy $S_{\bar{x}}^*$ is the optimal strategy satisfying (IA.9). Since the firm can also use the strategy $S_{\bar{x}}^* = (T^{\mathcal{D}}, \{N_t, U_t; t \in (0, T^{\mathcal{D}})\})$ when creditors' belief is $S_{\bar{x}}$ and equity issuance is costless, and obtain the same state processes, we have

$$\begin{aligned} \mathcal{P}^{\text{costless}}(Y, X; S_{\bar{x}}^*, S_{\bar{x}}) &= \mathbb{E}_t \left[\int_t^{T^{\mathcal{D}}} e^{-r(s-t)} (dU_s - dN_s) \right] \\ &\geq \mathbb{E}_t \left[\int_t^{T^{\mathcal{D}}} e^{-r(s-t)} (dU_s - dN_s - dH_s) \right] = \mathcal{P}(Y, X; S_{\bar{x}}^*, S_{\bar{x}}), \end{aligned} \quad (\text{IA.11})$$

where $\mathcal{P}^{\text{costless}}$ is the equity value with costless equity issuance.

Let

$$P^{\text{costless}}(Y, X; S_{\bar{x}}) = \max_{\hat{S}} \mathcal{P}^{\text{costless}}(Y, X; \hat{S}, S_{\bar{x}}), \quad \text{and} \quad p^{\text{costless}}(x; \bar{x}) = P^{\text{costless}}(1, x; S_{\bar{x}}).$$

Then we have $P^{\text{costless}}(Y, X; S_{\bar{x}}) \geq \mathcal{P}^{\text{costless}}(Y, X; S_{\bar{x}}^*, S_{\bar{x}})$. Combining with (IA.11), and using $p(x; \bar{x}) = \mathcal{P}(Y, X; S_{\bar{x}}^*, S_{\bar{x}})/Y$, we have $p^{\text{costless}}(x; \bar{x}) \geq p(x; \bar{x})$. Let $v^{\text{costless}}(\bar{x}) = \inf\{x > 0 : p^{\text{costless}}(x; \bar{x}) = 0\}$ and recall that $\mathcal{A}(\bar{x}) = \inf\{x > 0 : p(x; \bar{x}) = 0\}$. Then we have $v^{\text{costless}}(\bar{x}) \geq \mathcal{A}(\bar{x})$.

Using Lemma 3, we have $P^{\text{costless}}(1, 0; S_{\bar{x}}) = v^{\text{costless}}(\bar{x}) < \bar{x}$ when $\bar{x} > \frac{1}{r-g}$. Using Lemma 4, $v^{\text{costless}}(\bar{x})$ is increasing in $\bar{x} \geq \ell$. Then for any $\epsilon > 0$,

$$v^{\text{costless}}\left(\frac{1}{r-g}\right) \leq v^{\text{costless}}\left(\frac{1}{r-g} + \epsilon\right) < \frac{1}{r-g} + \epsilon. \quad (\text{IA.12})$$

Sending ϵ to 0 in (IA.12), we obtain $v^{\text{costless}}\left(\frac{1}{r-g}\right) \leq \frac{1}{r-g}$. Consequently,

$$\mathcal{A}(\bar{x}) \leq v^{\text{costless}}(\bar{x}) \leq \bar{x}, \quad \forall \bar{x} \geq \frac{1}{r-g}, \quad (\text{IA.13})$$

$$\mathcal{A}(\bar{x}) \leq v^{\text{costless}}(\bar{x}) < \bar{x}, \quad \forall \bar{x} > \frac{1}{r-g}. \quad (\text{IA.14})$$

Using the results proved in step 1, for any $\bar{x} \geq \ell$ and $x \geq 0$, we have

$$p(x; \bar{x}) \leq p(x; \bar{x}_1) \leq p^{\text{costless}}(x; \bar{x}_1) \leq v^{\text{costless}}(\bar{x}_1) \leq \bar{x}_1, \quad (\text{IA.15})$$

where $\bar{x}_1 = \max\{\bar{x}, \frac{1}{r-g}\}$, the last inequality uses (IA.13).

Step 3: We prove that there exists $\bar{x} \in [\bar{x}_0, \bar{x}_1)$, such that $\mathcal{A}(\bar{x}) = \bar{x}$ if $\mathcal{A}(\bar{x}_0) \geq \bar{x}_0$ and $\mathcal{A}(\bar{x}_1) < \bar{x}_1$ for some $\bar{x}_1 > \bar{x}_0 \geq \ell$.

Let $\mathbb{A} = \{x \in [\bar{x}_0, \bar{x}_1] : \mathcal{A}(x) \geq x\}$. Since $\mathcal{A}(\bar{x}_0) \geq \bar{x}_0$, $\mathbb{A}$ is nonempty.

Let $\bar{x} = \sup \mathbb{A}$. We have $\bar{x}_0 \leq \bar{x} \leq \bar{x}_1$. If $\bar{x} \notin \mathbb{A}$, then $\mathcal{A}(\bar{x}) < \bar{x}$. By the definition of $\bar{x} = \sup \mathbb{A}$ and $\bar{x} \notin \mathbb{A}$, there exists an increasing sequence $x_n \in \mathbb{A}$ such that $\lim_{n \rightarrow \infty} x_n = \bar{x}$. Thus $\mathcal{A}(x_n) \geq x_n$. Because $\mathcal{A}(x)$ is nondecreasing in x , we have

$$\mathcal{A}(\bar{x}) \geq \mathcal{A}(x_n) \geq x_n. \quad (\text{IA.16})$$

Sending n to infinity in the above inequality, we obtain $\mathcal{A}(\bar{x}) \geq \lim_{n \rightarrow \infty} x_n = \bar{x}$. Then we arrive at a contradiction.

Thus $\bar{x} \in \mathbb{A}$, and then $\mathcal{A}(\bar{x}) \geq \bar{x}$. Since $\mathcal{A}(\bar{x}_1) < \bar{x}_1$, we have $\bar{x} < \bar{x}_1$. By the definition of $\bar{x} = \sup \mathbb{A}$, for any small $\varepsilon \in (0, \bar{x}_1 - \bar{x})$, we have $\bar{x} + \varepsilon \notin \mathbb{A}$ and then $\mathcal{A}(\bar{x} + \varepsilon) < \bar{x} + \varepsilon$. Because $\mathcal{A}(x)$ is nondecreasing in x , we have

$$\bar{x} \leq \mathcal{A}(\bar{x}) \leq \mathcal{A}(\bar{x} + \varepsilon) < \bar{x} + \varepsilon. \quad (\text{IA.17})$$

Sending ε to 0 in the above inequality, we obtain $\mathcal{A}(\bar{x}) = \bar{x}$, and $\bar{x} < \bar{x}_1$.

Since $p(x; \bar{x}) \geq \max\{\ell - x, 0\} > 0$ for any $x < \ell$, we have $\mathcal{A}(\ell) \geq \ell$, where $p(x; \bar{x})$ is given in (IA.9). Using (IA.14) and the result in step 3, for any $\bar{x}_1 > \frac{1}{r-g}$, there exists $\bar{x} \in [\ell, \bar{x}_1)$, such that $\mathcal{A}(\bar{x}) = \bar{x}$, which implies that $\bar{x} = \inf\{x > 0 : p(x; \bar{x}) = 0\}$. It follows that $\bar{x}$ is the default boundary of an equilibrium strategy and $p(x; \bar{x})$ is the equilibrium (scaled) equity value.

Step 4: We show that the firm's equilibrium equity value $P(Y, X)$ is strictly increasing in Y and strictly decreasing in X for any $Y > 0$ and $X \in [0, \bar{x}Y]$.

Let $\bar{x}$ denote the default boundary of an equilibrium strategy. Using Proposition 1, equilibrium value $P(Y, X) > 0$ for any $Y > 0$ and $X \in [0, \bar{x}Y)$. For any $Y > 0$ and $X \in [0, \bar{x}Y)$, there exists a small $\varepsilon_0 > 0$, such that $(X + \varepsilon_0)/Y < \bar{x}$ and $P(Y, X + \varepsilon) > 0$ for any $\varepsilon \in [0, \varepsilon_0]$. Since paying dividend ε is a feasible suboptimal strategy, we derive from the optimality of the strategy S^* that

$$P(Y, X) \geq P(Y, X + \varepsilon) + \varepsilon > P(Y, X + \varepsilon).$$

Thus, $P(Y, X)$ is strictly decreasing in X for any $Y > 0$ and $X \in [0, \bar{x}Y]$.

Using Proposition 1, we have $P(Y, X) = YP(1, X/Y)$. For any $Y_1 > Y_2 > 0$ and $X \in [0, \bar{x}Y_2]$, we have

$$P(Y_1, X) = Y_1 P(1, X/Y_1) > Y_2 P(1, X/Y_1) \geq Y_2 P(1, X/Y_2) = P(Y_2, X).$$

Thus, $P(Y, X)$ is strictly increasing in Y . ■

Proof of Theorem 1. It suffices to show that, for every Markov strategy $S \in \Phi(S^*)$,

$$P(Y, X) \geq \mathcal{P}(Y, X; S, S^*), \quad \forall Y > 0, X \geq 0. \quad (\text{IA.18})$$

Let $S = (T^{\mathcal{D}}, \{N_t, U_t; t \in (0, T^{\mathcal{D}})\})$. Since U_t , N_t and H_t are nondecreasing processes, we have the following decomposition:

$$U_t = U_t^c + \sum_{s \in [0, t]} \Delta U_s, \quad N_t = N_t^c + \sum_{s \in [0, t]} \Delta N_s, \quad H_t = H_t^c + \sum_{s \in [0, t]} \Delta H_s,$$

where U_t^c , N_t^c , and H_t^c denote the continuous components of U_t , N_t , and H_t , respectively, and $\Delta U_s := U_s - U_{s-}$, $\Delta N_s := N_s - N_{s-}$, and $\Delta H_s := H_s - H_{s-}$ denote their jumps at time s . We normalize $U_{0-} = U_{0-}^c = N_{0-} = N_{0-}^c = H_{0-} = H_{0-}^c = 0$. According to (9), we have the following decomposition for X :

$$X_t = X_t^c + \sum_{s \in [0, t]} \Delta X_s,$$

where $X_t^c = X_{0-} + \int_0^t (C_s + \Theta_s - Y_s) ds + U_t^c - N_t^c$ denotes the continuous component of X_t , $\Delta X_t := X_t - X_{t-} = \Delta U_t - \Delta N_t$ denotes the jump of X_t , X_{0-} is the firm's initial debt balance, C_s is the interest payment, and Θ_s is the tax payment.

First, when equity issuance is costly, it is suboptimal for the firm to make a payout and issue equity at the same time. Thus, without loss of generality, it suffices to consider strategies for which $\Delta U_t \Delta N_t = 0$ for all t .⁴⁶ Second, at the default time $t = T^{\mathcal{D}}$, the firm is not allowed to make a payout as creditors have absolute priority over equity holders. Thus, we only consider the strategy that satisfies $\Delta U_{T^{\mathcal{D}}} = 0$. Third, the firm is unable to issue equity at the default time $t = T^{\mathcal{D}}$ due to equity holders' limited liability. Thus, we only consider the strategy that satisfies $\Delta N_{T^{\mathcal{D}}} = 0$.

Since $S \in \Phi(S^*)$, $(Y_t, X_t) \notin \mathcal{D}^*$ for almost every $t \in [0, T^{\mathcal{D}}]$. Then $(Y_t, X_t) \in \mathbb{Z} = \{(Y, X) | 0 < Y < \infty, 0 \leq X \leq \bar{x}Y\}$, and (9) holds for almost every $t \in [0, T^{\mathcal{D}}]$. Moreover, the credit spread (4) is given by $\eta_{t-} = \eta(X_{t-}/Y_{t-}; \bar{x})$, where $\eta(x; \bar{x})$ is given in (A.7). Recall that $p(x; S^*) = \mathcal{P}(1, x; S^*, S^*) = P(1, x)$ and note that $p(x; S^*)$ and $p_x(x; S^*)$ are bounded on the compact set $x \in [0, \bar{x}]$. Since $P(Y, X) = Yp(X/Y; S^*)$, we have that $P_X(Y, X) = p_x(X/Y; S^*)$ and $P_Y(Y, X) = p(X/Y; S^*) - X/Y p_x(X/Y; S^*)$ are bounded in $\mathbb{Z} = \{(Y, X) | 0 < Y < \infty, 0 \leq X \leq \bar{x}Y\}$.

⁴⁶ If $\Delta U_t \geq \Delta N_t$, then it is better for the firm to payout $\Delta U_t - \Delta N_t$ without issuing equity. If $\Delta U_t < \Delta N_t$, then it is better for the firm to issue equity $\Delta N_t - \Delta U_t$ without making a payout. When equity issuance is costless, we use $U_t - N_t$ as control variable, and set $dU_t dN_t = 0$ without loss of generality. See Internet Appendix IB for the details.

Using the generalized Itô's formula and omitting the subscripts t and $t-$ on the state processes (Y_t, X_t) , we can write:

$$\begin{aligned}
& dP(Y, X) \\
&= \left[\mu Y P_Y(Y, X) + \frac{\sigma^2 Y^2}{2} P_{YY}(Y, X) \right] dt + P_Y(Y, X) \sigma Y d\mathcal{B}_t \\
&\quad + [P(ZY, X) - P(Y, X)] d\mathcal{J}_t + P_X(Y, X) dX_t^c + P(Y, X + \Delta X_t) - P(Y, X) \\
&= \left[\mu Y P_Y(Y, X) + \frac{\sigma^2 Y^2}{2} P_{YY}(Y, X) + (C + \Theta - Y) P_X(Y, X) + \lambda [P(ZY, X) - P(Y, X)] \right] dt \\
&\quad + P_Y(Y, X) \sigma Y d\mathcal{B}_t + [P(ZY, X) - P(Y, X)] (d\mathcal{J}_t - \lambda dt) \\
&\quad + P_X(Y, X) dX_t^c - (C + \Theta - Y) P_X(Y, X) dt + P(Y, X + \Delta X_t) - P(Y, X),
\end{aligned}$$

where $\Theta = \theta(c)Y$ is the firm's tax payment (with $\theta(c)$ given in (5) and $c = C/Y$), $C = (r + \eta(X/Y; \bar{x}))X$ is the interest payment, and $\eta(x; \bar{x})$ is the credit spread given in (A.7).

Using $dX_t^c = (C_t + \Theta_t - Y_t)dt + dU_t^c - dN_t^c$, we derive from above that

$$\begin{aligned}
& d(e^{-rt} P(Y, X)) \\
&= e^{-rt} \tilde{\mathcal{L}}P(Y, X)dt + e^{-rt} P_Y(Y, X) \sigma Y d\mathcal{B}_t + e^{-rt} [P(ZY, X) - P(Y, X)] (d\mathcal{J}_t - \lambda dt) \\
&\quad + e^{-rt} [P_X(Y, X)(dU_t^c - dN_t^c) + P(Y, X + \Delta X_t) - P(Y, X)], \tag{IA.19}
\end{aligned}$$

where

$$\tilde{\mathcal{L}}P = \mu Y P_Y + \frac{\sigma^2 Y^2}{2} P_{YY} + (C + \Theta - Y) P_X + \lambda [P(ZY, X) - P(Y, X)] - rP. \tag{IA.20}$$

Denote

$$T_n = \inf\{u \in [t, T^{\mathcal{D}}] : \int_t^u |e^{-r(s-t)} Y_s|^2 ds > n\} \tag{IA.21}$$

with $\inf \emptyset = T^{\mathcal{D}}$. Integrating equation (IA.19) from t to T_n and using $\Delta X = \Delta U$ or $-\Delta N$, we obtain

$$\begin{aligned}
& e^{-r(T_n-t)} P(Y_{T_n}, X_{T_n}) \\
&= P(Y_t, X_{t-}) + \int_t^{T_n} e^{-r(s-t)} \tilde{\mathcal{L}}P(Y_s, X_s) ds \\
&\quad + \int_t^{T_n} e^{-r(s-t)} P_Y(Y_s, X_s) \sigma Y_s d\mathcal{B}_s + \int_t^{T_n} e^{-r(s-t)} [P(ZY_{s-}, X_{s-}) - P(Y_{s-}, X_{s-})] (d\mathcal{J}_s - \lambda ds) \\
&\quad + \int_t^{T_n} e^{-r(s-t)} P_X(Y_s, X_s) dU_s^c + \sum_{t \leq s \leq T_n} e^{-r(s-t)} (P(Y_s, X_{s-} + \Delta U_s) - P(Y_s, X_{s-})) \\
&\quad + \int_t^{T_n} e^{-r(s-t)} P_X(Y_s, X_s) (-dN_s^c) + \sum_{t \leq s \leq T_n} e^{-r(s-t)} (P(Y_s, X_{s-} - \Delta N_s) - P(Y_s, X_{s-})). \tag{IA.22}
\end{aligned}$$

Since $P_Y(Y, X)$ is bounded in the region $\mathbb{Z}$ and $(Y_s, X_s) \in \mathbb{Z}$, the third term in (IA.22), the integral with respect to $d\mathcal{B}_s$, is a square integrable martingale for $s \in [t, T_n]$. Similarly, since $P(Y, X) = Yp(X/Y; S^*)$ and $p(x; S^*)$ is bounded, the fourth term in (IA.22), the integral with respect to $(d\mathcal{J}_s - \lambda ds)$, is also a square integrable martingale for $s \in [t, T_n]$. Taking expectations in (IA.22), we obtain:

$$\begin{aligned} P(Y_t, X_{t-}) &= \mathbb{E}_t \left[e^{-r(T_n-t)} P(Y_{T_n}, X_{T_n}) \right] - \mathbb{E}_t \left[\int_t^{T_n} e^{-r(s-t)} \tilde{\mathcal{L}} P(Y_s, X_s) ds \right] \\ &+ \mathbb{E}_t \left[- \int_t^{T_n} e^{-r(s-t)} P_X(Y_s, X_s) dU_s^c - \sum_{t \leq s \leq T_n} e^{-r(s-t)} (P(Y_s, X_{s-} + \Delta U_s) - P(Y_s, X_{s-})) \right] \\ &+ \mathbb{E}_t \left[\int_t^{T_n} e^{-r(s-t)} P_X(Y_s, X_s) dN_s^c + \sum_{t \leq s \leq T_n} e^{-r(s-t)} (P(Y_s, X_{s-}) - P(Y_s, X_{s-} - \Delta N_s)) \right]. \end{aligned} \quad (\text{IA.23})$$

Since $S \in \Phi(S^*)$, we observe from Definition 2-(i) that for $t < T^\mathcal{D}$ such that $\Delta U_t > 0$, we have $X_{t-} \leq X_t + \Delta U_t \leq \bar{x}Y_t$. Combining with $P_X(Y, X) \leq -1$ for any $X \leq \bar{x}Y$, we conclude that

$$\begin{aligned} &\mathbb{E}_t \left[- \int_t^{T_n} e^{-r(s-t)} P_X(Y_s, X_s) dU_s^c - \sum_{t \leq s \leq T_n} e^{-r(s-t)} (P(Y_s, X_{s-} + \Delta U_s) - P(Y_s, X_{s-})) \right] \\ &\geq \mathbb{E}_t \left[\int_t^{T_n} e^{-r(s-t)} dU_s^c + \sum_{t \leq s \leq T_n} e^{-r(s-t)} \Delta U_s \right] = \mathbb{E}_t \left[\int_{t \leq s \leq T_n} e^{-r(s-t)} dU_s \right]. \end{aligned} \quad (\text{IA.24})$$

Since $P_X(Y, X) \leq -1$ for any $X < \bar{x}Y$ and $P(Y, X) = 0$ for any $X \geq \bar{x}Y$, $P(Y, X)$ is non-increasing in X , which implies that $P^N(Y, X) = \sup_{M>0} P(Y, X - M) - h_0Y - (1 + h_1)M$ is non-increasing in X . For any $X \geq \bar{x}Y$, we have

$$P^N(Y, X) \leq P^N(Y, \bar{x}Y) \leq P(Y, \bar{x}Y) = 0,$$

where the second inequality uses $P(Y, X) \geq P^N(Y, X)$ for any $(Y, X) \in \mathbb{Z}$. Thus, $P(Y, X) \geq P^N(Y, X)$ for any $Y > 0$ and $X \geq 0$.

Using $P(Y, X) \geq P^N(Y, X)$, we have $P(Y_s, X_{s-}) \geq P(Y_s, X_{s-} - \Delta N_s) - h_0Y_s - (1 + h_1)\Delta N_s = P(Y_s, X_{s-} - \Delta N_s) - \Delta N_s - \Delta H_s$. It follows that

$$P(Y_s, X_{s-}) - P(Y_s, X_{s-} - \Delta N_s) \geq -(\Delta N_s + \Delta H_s). \quad (\text{IA.25})$$

In the case $h_0 > 0$, we must have $dN_s^c = 0$ for almost every s . Otherwise, cumulative equity issuance costs are infinitely large, which violates condition 3.) of Definition 1. In the case $h_0 = 0$, we have $P(Y, X) \geq P^N(Y, X) \geq P(Y, X - \varepsilon) - (1 + h_1)\varepsilon$ for any small $\varepsilon > 0$, which yields that

$P_X(Y, X) \geq -1 - h_1$ for $(Y, X) \in \mathbb{Z}$. Hence, we have

$$\begin{aligned}
& \mathbb{E}_t \left[\int_t^{T_n} e^{-r(s-t)} P_X(Y_s, X_s) dN_s^c + \sum_{t \leq s \leq T_n} e^{-r(s-t)} (P(Y_s, X_{s-}) - P(Y_s, X_{s-} - \Delta N_s)) \right] \\
& \geq \mathbb{E}_t \left[\mathbf{1}_{h_0=0} \int_t^{T_n} e^{-r(s-t)} (-1 - h_1) dN_s^c - \sum_{t \leq s \leq T_n} e^{-r(s-t)} (\Delta N_s + \Delta H_s) \right] \\
& = \mathbb{E}_t \left[\mathbf{1}_{h_0=0} \int_t^{T_n} e^{-r(s-t)} (-dN_s^c - dH_s^c) - \sum_{t \leq s \leq T_n} e^{-r(s-t)} (\Delta N_s + \Delta H_s) \right] \\
& = - \mathbb{E}_t \left[\int_{t \leq s \leq T_n} e^{-r(s-t)} (dN_s + dH_s) \right], \tag{IA.26}
\end{aligned}$$

where the inequality follows from the fact that $(Y_s, X_s) \in \mathbb{Z}$ for almost every $s \in [0, T_n)$, from (IA.25), and, in the case $h_0 = 0$, from $P_X(Y, X) \geq -1 - h_1$ for $(Y, X) \in \mathbb{Z}$.

Substituting (IA.24)-(IA.26) into (IA.23), and using $P(Y, X) \geq 0$, $\mathbb{E}_t[\tilde{\mathcal{L}}P(Y_s, X_s)] = \mathbb{E}_t[\mathcal{L}P(Y_s, X_s)] \leq 0$ for almost every $s < T_n \leq T^\mathcal{D}$, we obtain

$$P(Y_t, X_{t-}) \geq \mathbb{E}_t \left[\int_{t \leq s \leq T_n} e^{-r(s-t)} (dU_s - dN_s - dH_s) \right]. \tag{IA.27}$$

Because processes U_s , N_s and H_s are nondecreasing in s , and because Definition 1 implies $\mathbb{E}_t \left[\int_t^{T^\mathcal{D}} e^{-r(s-t)} dU_s \right] < \infty$, $\mathbb{E}_t \left[\int_t^{T^\mathcal{D}} e^{-r(s-t)} (dN_s + dH_s) \right] < \infty$, the monotone convergence theorem implies

$$\lim_{n \rightarrow \infty} \mathbb{E}_t \left[\int_{t \leq s \leq T_n} e^{-r(s-t)} (dU_s - dN_s - dH_s) \right] = \mathbb{E}_t \left[\int_{t \leq s \leq T^\mathcal{D}} e^{-r(s-t)} (dU_s - dN_s - dH_s) \right]. \tag{IA.28}$$

Sending n to infinity in (IA.27), and using (IA.28), we obtain

$$P(Y_t, X_{t-}) \geq \mathbb{E}_t \left[\int_{t \leq s \leq T^\mathcal{D}} e^{-r(s-t)} (dU_s - dN_s - dH_s) \right] = \mathcal{P}(Y_t, X_{t-}; S, S^*). \tag{IA.29}$$

This proves (IA.18). ■

Proof of Theorem 2. Recall the definition of $\mathcal{A}(\bar{x})$ in (IA.10), and denote $\bar{\mathbb{X}} = \{\bar{x} \geq \ell : \mathcal{A}(\bar{x}) = \bar{x}\}$. For any MPE $S \in \mathcal{S}_x$, we have $S = \arg \max_{\hat{S}} \mathcal{P}(Y, X; \hat{S}, S)$. Moreover, from Proposition 1, it follows that the default region in the strategy S is given by $\{(Y, X) : X/Y > \bar{x}\}$, where $\bar{x} = \inf\{x > 0 : p(x; \bar{x}) = 0\}$, and $p(x; \bar{x}) = \mathcal{P}(1, x; S, S)$.

We shall denote the equilibrium strategy with default region $\{(Y, X) : X/Y > \bar{x}\}$ as $S_{\bar{x}}$, and the corresponding equity value as $p(x; \bar{x})$. Then, all equilibria in $\mathcal{S}_x$ can be represented by $\{S_{\bar{x}} : \bar{x} \in \bar{\mathbb{X}}\}$. For any $\bar{x}_1, \bar{x}_2 \in \bar{\mathbb{X}}$ with $\bar{x}_1 \leq \bar{x}_2$, we derive from the result in step 1 in the proof of Proposition 2 that $p(x; \bar{x}_1) \leq p(x; \bar{x}_2)$ for all x . Thus, we only need to show that there exists $\bar{x}^* \in \bar{\mathbb{X}}$, such that $\bar{x}^* \geq \bar{x}$ for any $\bar{x} \in \bar{\mathbb{X}}$.

According to the result in step 2 in the proof of Proposition 2, we have $\mathcal{A}(\bar{x}) < \bar{x}$ for all $\bar{x} > \frac{1}{r-g}$. Thus, $\mathcal{A}(\bar{x}) = \bar{x} \leq \frac{1}{r-g}$ for any $\bar{x} \in \bar{\mathbb{X}}$. Let $\bar{x}^* = \sup \bar{\mathbb{X}}$. We have $\bar{x}^* \leq \frac{1}{r-g}$. We only need to prove that $\bar{x}^* \in \bar{\mathbb{X}}$. For the sake of contradiction, assume that $\bar{x}^* \notin \bar{\mathbb{X}}$. There exists an increasing sequence $x_n \in \bar{\mathbb{X}}$ such that $\lim_{n \rightarrow \infty} x_n = \bar{x}^*$. Thus $\mathcal{A}(x_n) = x_n$. Because $\mathcal{A}(x)$ is nondecreasing in x , we have

$$\mathcal{A}(\bar{x}^*) \geq \mathcal{A}(x_n) = x_n. \quad (\text{IA.30})$$

Sending n to infinity in the above inequality, we obtain $\mathcal{A}(\bar{x}^*) \geq \lim_{n \rightarrow \infty} x_n = \bar{x}^*$. Since $\bar{x}^* \notin \bar{\mathbb{X}}$, we have $\mathcal{A}(\bar{x}^*) > \bar{x}^*$. Using the result in step 2 in the proof of Proposition 2, we have $\mathcal{A}(\bar{x}_1) < \bar{x}_1$ for $\bar{x}_1 > \max\{\frac{1}{r-g}, \bar{x}^*\}$. Therefore, we conclude from the result in step 3 in the proof of Proposition 2 that there exists $\bar{x}_2 \in (\bar{x}^*, \bar{x}_1)$, such that $\mathcal{A}(\bar{x}_2) = \bar{x}_2$. It follows that $\bar{x}_2 \in \bar{\mathbb{X}}$. By the definition of $\bar{x}^* = \sup \bar{\mathbb{X}}$, we arrive at a contradiction as $\bar{x}_2 > \bar{x}^*$. Therefore, $\bar{x}^* \in \bar{\mathbb{X}}$. ■

In order to prove Theorem 3, we first prove the following lemma, which gives the variational inequality for the optimal equity value given the debtholders' conjectured default region $\{(Y, X) : X/Y > \bar{x}\}$.

Lemma 2. *Consider a Markov strategy $\tilde{S} \in \mathcal{S}_x$ with default region $\tilde{\mathcal{D}} = \{(Y, X) : X/Y > \bar{x}\}$ for some $\bar{x} \geq \ell$. Suppose a Markov strategy $S^* \in \mathcal{S}_x$ satisfies $\mathcal{P}(Y, X; S^*, \tilde{S}) = \max_{\hat{S}} \mathcal{P}(Y, X; \hat{S}, \tilde{S})$ for any $(Y, X) \in \Sigma$. Denote $\tilde{p}(x) = \mathcal{P}(1, x; S^*, \tilde{S})$ and $v = \inf\{x > 0 : \tilde{p}(x) = 0\}$. We have*

$$\max\{\tilde{p}^N(x) - \tilde{p}(x), \max\{\ell - x, 0\} - \tilde{p}(x)\} = 0, \quad x > \min\{v, \bar{x}\}. \quad (\text{IA.31})$$

Moreover, if $v \geq \bar{x}$, then $\tilde{p}(x)$ is the unique continuous and bounded viscosity solution to the following equation for $x \in [0, \bar{x}]$.⁴⁷

$$\max\{\mathcal{L}^{\bar{x}}\tilde{p}(x), 1 + \tilde{p}_x(x), \tilde{p}^N(x) - \tilde{p}(x)\} = 0, \quad (\text{IA.32})$$

where $\mathcal{L}^{\bar{x}}$ is given in (B.7), and $\tilde{p}^N(x) = \max_{m>0} \tilde{p}(x - m) - h_0 - (1 + h_1)m$.

Proof of Lemma 2. In the following, we denote equity value $\tilde{P}(Y, X) = \mathcal{P}(Y, X; S^*, \tilde{S})$.

Because default and equity financing are feasible suboptimal strategies, we derive from the optimality of strategy S^* that

$$\tilde{P}(Y, X) \geq \max\{\ell Y - X, 0\}, \quad (\text{IA.33})$$

$$\tilde{P}(Y, X) \geq \tilde{P}^N(Y, X), \quad (\text{IA.34})$$

for any $(Y, X) \in \Sigma$, where $\tilde{P}^N(Y, X) = \sup_{M>0} \tilde{P}(Y, X - M) - h_0 Y - (1 + h_1)M$. Using (B.2) and $\tilde{p}(x) = \tilde{P}(1, x)$, we have $\tilde{P}(Y, X) = Y\tilde{p}(X/Y)$. Then we derive from (IA.33)-(IA.34) that

$$\tilde{p}(x) \geq \max\{\ell - x, 0\}, \quad (\text{IA.35})$$

$$\tilde{p}(x) \geq \tilde{p}^N(x), \quad (\text{IA.36})$$

⁴⁷ Boundary conditions are $\tilde{p}_x(0) = -1$ and (IA.31) for $x = \bar{x}$.

for any $x \geq 0$. For $x > \bar{x}$, the firm can only default or purchase debt back such that firm leaves default region immediately. Therefore, we can see that (IA.31) holds for $x > \bar{x}$. Using Proposition 1, for any $x > v$, we have $\tilde{p}(x) = 0 \geq \max\{\ell - x, 0\}$ and thus $\ell - v \leq 0$. Therefore, we can see that (IA.31) holds for $x \geq v$ with $\tilde{p}(x) = 0 = \max\{\ell - x, 0\}$.

According to (IA.15), we can see that $\tilde{p}(x) \leq \max\{\bar{x}, \frac{1}{r-g}\}$. According to (IA.35), we have $\tilde{p}(x) \geq 0$. Thus, $\tilde{p}(x)$ is bounded.

Equation (IA.32) is derived using dynamic programming and (B.2); the technical arguments are similar to those in Fleming and Soner (2006) and Seydel (2009). Moreover, the uniqueness and continuity of the viscosity solution to (IA.32) can be established by adapting the arguments in Seydel (2009). ■

Proof of Theorem 3. From the homogeneity property (B.2), for any MPE $S^* \in \mathcal{S}_x$, the scaled equity value $p(x) = p(x; S^*)$ and equity value $P(Y, X) = \mathcal{P}(Y, X; S^*, S^*)$ satisfy $P(Y, X) = Yp(X/Y)$.

Since S^* is an MPE, $\mathcal{P}(Y, X; S^*, S^*) \geq \mathcal{P}(Y, X; \hat{S}, S^*)$ for all $(Y, X) \in \Sigma$ and $\hat{S} \in \Phi(S^*)$. Then it follows from Proposition 1 that $p(x) = 0$ for any $x \geq \bar{x}$, and $\mathcal{D} = \{(Y, X) : X/Y > \bar{x}\}$ is the default region under the strategy S^* , where $\bar{x} = \inf\{x > 0 : p(x) = 0\} \geq \ell$. Combining this with Lemma 2, we conclude that $p(x)$ is the unique bounded viscosity solution to the equation (B.6).

From the proof of Theorem 2, we have $\bar{x}^* = \sup \bar{\mathbb{X}} \leq \frac{1}{r-g}$ and $\bar{x}^* \in \bar{\mathbb{X}}$, where $\bar{\mathbb{X}} = \{\bar{x} \geq \ell : \mathcal{A}(\bar{x}) = \bar{x}\}$ is the set of default boundaries $\bar{x}$ for all MPEs in the set $\mathcal{S}_x$, and $\mathcal{A}(\bar{x})$ is given in (IA.10).

Let $p(x; \bar{x})$ denote the scaled equity value for an MPE in the set $\mathcal{S}_x$ with the default boundary $\bar{x} \geq \ell$. For any $\bar{x}_1, \bar{x}_2 \in \bar{\mathbb{X}}$ with $\bar{x}_1 \leq \bar{x}_2$, it follows from the result in step 1 in the proof of Proposition 2 that $p(x; \bar{x}_1) \leq p(x; \bar{x}_2)$ for all x . Hence, the MPE with default boundary $\bar{x}^*$ is the best MPE.

In addition, absent equity issuance at $\bar{x}$, diffusion shocks to x_t would trigger default almost surely. When $\bar{x} > \ell$, the fractional loss given default is strictly positive, preventing creditors from breaking even and thus yielding a contradiction.⁴⁸ Therefore, if $\bar{x} > \ell$, the firm must issue equity at $\bar{x}$ and $p^N(\bar{x}) = p(\bar{x}) = 0$. ■

Proof of Proposition 3. In the case $h_0 = h_1 = 0$, the characterization and uniqueness of equilibrium are proved in Appendix IB.3. Thus, we only need to consider the case $h_0 > 0$ or $h_1 > 0$.

Since $(1 - \tau)(1 - \frac{\ell}{\bar{x}})zF'(z) - \tau F(z)$ is strictly increasing in $z \in [0, 1]$, $F(z)$ is increasing in $z \in [0, 1]$, and $\bar{x} \geq \ell$, it follows that $\bar{x} > \ell$. For the sake of contradiction, assume that $p^N(\bar{x}) < p(\bar{x})$. Then the firm does not issue equity at $\bar{x}$. Starting from $\bar{x}$, diffusion shocks to x_t then lead to default almost surely. Since $\bar{x} > \ell$, the fractional loss given default is strictly positive.

⁴⁸ If the firm does not issue equity at $\bar{x}$, then $T^D = t$ when $x_t = \bar{x}$, contradicting with Definition 1-(4a).

In this case, creditors cannot break even, which yields a contradiction. Therefore, $p^N(\bar{x}) = p(\bar{x})$ and the firm must issue equity at $\bar{x}$. See also footnote 48 for an alternative argument.

Fix an MPE with default boundary $\bar{x}$ and let $p(x)$ be its scaled equity value. By Theorem 3, $p(x)$ is the unique continuous and bounded viscosity solution of the equation (B.6) on $[0, \bar{x}]$ with $p(x) = 0$ for any $x \geq \bar{x}$. Let $\underline{x} > 0$ denote the right endpoint of the highest payout boundary. Since the firm issues equity at $\bar{x}$, and since $h_0 > 0$ or $h_1 > 0$, it is not optimal to make a payout and issue equity simultaneously. Therefore, $\underline{x} < \bar{x}$. Moreover, there exists $\underline{x}_1 < \underline{x}$, such that the firm makes payouts for any $x \in [\underline{x}_1, \underline{x}]$. Therefore, $p(x) = p(\underline{x}) + \underline{x} - x$ for any $x \in [\underline{x}_1, \underline{x}]$.

We define an affine-pasted function $\tilde{p}(x)$ as follows:

$$\tilde{p}(x) = p(\underline{x}) + \underline{x} - x, \quad x \leq \underline{x}, \quad (\text{IA.37})$$

$$\tilde{p}(x) = p(x), \quad x > \underline{x}. \quad (\text{IA.38})$$

We only need to show that $\tilde{p}(x)$ also satisfies equation (B.6) on $[0, \underline{x}]$. Indeed, p and $\tilde{p}$ are two continuous bounded viscosity solutions of (B.6) on $[0, \underline{x}]$ with the same exterior data for $x \geq \underline{x}$. The comparison principle underlying the uniqueness result in Theorem 3 implies that $\tilde{p}(x) = p(x)$ on $[0, \underline{x}]$. Hence, the firm makes payouts for all $x \leq \underline{x}$.

Since $\tilde{p}_x(x) = -1$ for $x < \underline{x}$, to prove that $\tilde{p}(x)$ satisfies equation (B.6) on $[0, \underline{x}]$, it remains to verify that, for every $x \in [0, \underline{x}]$, $\mathcal{L}^{\tilde{p}}\tilde{p}(x) < 0$ and, for every $m > 0$, $\tilde{p}(x - m) - h_0 - (1 + h_1)m < \tilde{p}(x)$. For $m > 0$ and any $x \leq \underline{x}$, we have

$$\begin{aligned} \tilde{p}(x - m) - h_0 - (1 + h_1)m &= p(\underline{x}) + \underline{x} - (x - m) - h_0 - (1 + h_1)m \\ &= p(\underline{x}) + \underline{x} - x - h_0 - h_1m = \tilde{p}(x) - h_0 - h_1m < \tilde{p}(x), \end{aligned} \quad (\text{IA.39})$$

where the inequality follows from $h_0 > 0$ or $h_1 > 0$. The remainder of the proof that $\mathcal{L}^{\tilde{p}}\tilde{p}(x) < 0$, $x \in [0, \underline{x}]$ is organized into three steps.

Step 1: We show that there exists $\underline{x}_2 \in (\underline{x}, \bar{x})$, such that it is not optimal to issue equity for any $x \in [\underline{x}, \underline{x}_2]$.

Since the firm can not make a payout immediately before default, we have $\underline{x} < \bar{x}$. We first consider the case $h_0 > 0$. By the continuity and monotonicity of $p(x)$, there exists $\epsilon > 0$, such that $p(x) \in [p(\underline{x}) - h_0/2, p(\underline{x})]$ for any $x \in [\underline{x}, \underline{x} + \epsilon]$. Thus, for any $x \in [\underline{x}, \underline{x} + \epsilon]$ and any $m > 0$ such that $x - m \geq \underline{x}$, we have

$$p(x - m) - h_0 - (1 + h_1)m < p(x). \quad (\text{IA.40})$$

Moreover, since $p_x(x) = -1$ for any $x \in [\underline{x}_1, \underline{x}]$, (IA.40) also holds for any $x \in [\underline{x}, \underline{x} + \epsilon]$ and any $m > 0$ such that $x - m \geq \underline{x}_1$.

Suppose, toward a contradiction, that there exists a sequence $\{x_n\}_{n \geq 1}$, such that the firm optimally issues equity at $x_n > \underline{x}$, and $\lim_{n \rightarrow \infty} x_n = \underline{x}$. Since (IA.40) holds whenever $x - m \geq \underline{x}_1$, we must have

$$p(x_n) = \max_{x_n - m < \underline{x}_1} p(x_n - m) - h_0 - (1 + h_1)m = \max_{\tilde{x} < \underline{x}_1} p(\tilde{x}) - h_0 - (1 + h_1)(x_n - \tilde{x}). \quad (\text{IA.41})$$

Letting $n \rightarrow \infty$ and using the continuity of $p(x)$, we obtain

$$p(\underline{x}) = \max_{\tilde{x} < \underline{x}_1} p(\tilde{x}) - h_0 - (1 + h_1)(\underline{x} - \tilde{x}). \quad (\text{IA.42})$$

Then, for any $x \in [\underline{x}_1, \underline{x}]$, we have

$$\begin{aligned} & p(x) - [\max_{\tilde{x} < \underline{x}_1} p(\tilde{x}) - h_0 - (1 + h_1)(x - \tilde{x})] \\ &= p(x) - [p(\underline{x}) + (1 + h_1)(\underline{x} - x)] = -h_1(\underline{x} - x), \end{aligned} \quad (\text{IA.43})$$

where the first equality uses (IA.42), and the second equality uses $p_x(x) = -1$ for all $x \in [\underline{x}_1, \underline{x}]$. If $h_1 > 0$, then (IA.43) is strictly negative, contradicting the fact that $p(x)$ is the equilibrium equity value.

Hence, it remains only to derive a contradiction in the case $h_1 = 0$. Since $p(x)$ is the equilibrium equity value and the firm can always make a payout when $x < \bar{x}$, we have

$$p(x) \geq p(x + \Delta x) + \Delta x, \quad \Delta x \in [0, \bar{x} - x), \quad x < \bar{x}. \quad (\text{IA.44})$$

Recall that equity issuance is optimal at $x_n > \underline{x}$. By (IA.44), $p(\underline{x}) \geq p(x) + x - \underline{x}$ for any $x \in [\underline{x}, x_n]$. If there exists $x \in (\underline{x}, x_n)$, such that $p(\underline{x}) > p(x) + x - \underline{x}$, then

$$p(x_n) \leq p(x) + x - x_n < p(\underline{x}) - x + \underline{x} + x - x_n = p(\underline{x}) + \underline{x} - x_n, \quad (\text{IA.45})$$

where the first inequality follows from (IA.44). Comparing (IA.41) with (IA.42) yields $p(x_n) - p(\underline{x}) = (1 + h_1)(\underline{x} - x_n) = \underline{x} - x_n$ because $h_1 = 0$. This contradicts (IA.45). Therefore, $p(\underline{x}) = p(x) + x - \underline{x}$ for any $x \in (\underline{x}, x_n)$. Hence it is optimal for the firm to make payouts throughout $(\underline{x}, x_n)$, contradicting the definition of $\underline{x}$ as the right endpoint of the highest payout component.

Thus, when $h_0 > 0$, we have shown that there exists $\underline{x}_2 \in (\underline{x}, \bar{x})$, such that it is not optimal to issue equity for any $x \in [\underline{x}, \underline{x}_2]$. Next, consider the case $h_0 = 0$ and $h_1 > 0$. In this case, (B.6) becomes

$$\max \{ \mathcal{L}^{\bar{x}} p(x), 1 + p_x(x), -1 - h_1 - p_x(x) \} = 0, \quad x \in [0, \bar{x}], \quad (\text{IA.46})$$

and the equity financing region is given by $\{x \leq \bar{x} : 1 + h_1 + p_x(x) = 0\}$. Since the firm makes payouts in the region $x \in [\underline{x}_1, \underline{x}]$, we have $p_x(\underline{x}) = -1 > -1 - h_1$. Therefore, there exists $\underline{x}_2 \in (\underline{x}, \bar{x})$, such that it is not optimal to issue equity for any $x \in [\underline{x}, \underline{x}_2]$.⁴⁹

In sum, we have shown that there exists $\underline{x}_2 \in (\underline{x}, \bar{x})$, such that the firm neither issues equity nor makes a payout in the region $(\underline{x}, \underline{x}_2)$. In this region, the variational inequality (B.6) reduces to $\mathcal{L}^{\bar{x}} p(x) = 0$ for any $x \in (\underline{x}, \underline{x}_2)$, with the smooth-pasting condition (13) holding at $\underline{x}$. Hence, on $(\underline{x}, \underline{x}_2)$, the viscosity solution $p(x)$ is a classical solution.

⁴⁹ Intuitively, the equity financing region cannot be adjacent to the payout region: otherwise, the diffusion shocks in x_t would force the firm to alternate between issuing equity and making payouts, generating infinitely frequent interventions and hence an infinite expected cost due to the financing wedge $h_1 > 0$.

Step 2: We show that $\frac{d}{dx}\mathcal{L}^{\bar{x}}\tilde{p}(x)|_{x=\underline{x}} \geq 0$.

Recall that $p(x) = p(\underline{x}) + \underline{x} - x$ for any $x \in [\underline{x}_1, \underline{x}]$, and that $\mathcal{L}^{\bar{x}}p(x) \leq 0$ for $x \in [\underline{x}_1, \underline{x}]$ by Theorem 3. Moreover, since $\mathcal{L}^{\bar{x}}p(x) = 0$ for any $x \in (\underline{x}, \underline{x}_2)$, it follows that $\mathcal{L}^{\bar{x}}p(\underline{x}+) = 0 \geq \mathcal{L}^{\bar{x}}p(\underline{x}-)$. Since smooth-pasting condition (13) holds at $\underline{x}$, $p(x)$ and $p_x(x)$ are continuous at $x = \underline{x}$, $0 \leq \mathcal{L}^{\bar{x}}p(\underline{x}+) - \mathcal{L}^{\bar{x}}p(\underline{x}-) = \frac{\sigma^2 \underline{x}^2}{2}[p_{xx}(\underline{x}+) - p_{xx}(\underline{x}-)]$. Hence, $p_{xx}(\underline{x}+) \geq p_{xx}(\underline{x}-) = 0$. On the other hand, since $p_x(x) \leq -1$ for $x \in (\underline{x}, \underline{x}_2)$ and $p_x(\underline{x}) = -1$, it follows that $p_{xx}(\underline{x}+) \leq 0$. Therefore, $p_{xx}(\underline{x}+) = 0$ and thus $p_{xx}(x)$ is continuous at $x = \underline{x}$. Since $p(x)$, $p_x(x)$ and $p_{xx}(x)$ are continuous at $x = \underline{x}$, $\mathcal{L}^{\bar{x}}p(\underline{x}-) = \mathcal{L}^{\bar{x}}p(\underline{x}+) = 0$.

For any $x \leq \underline{x}$, using $\tilde{p}(x) = p(\underline{x}) + \underline{x} - x$, we have $\tilde{p}_x(x) = -1$. Therefore,

$$\mathcal{L}^{\bar{x}}\tilde{p}(x) = -A(x) + \lambda \left[\mathbb{E}[Z\tilde{p}(x/Z)] - \tilde{p}(x) \right] - (r - \mu)\tilde{p}(x), \quad (\text{IA.47})$$

where $A(x) = c(x; \bar{x}) + \theta(c(x; \bar{x})) - 1 - \mu x$. Since the firm makes a payout for every $x \in [\underline{x}_1, \underline{x}]$, we have

$$p(x) = p(\underline{x}) + \underline{x} - x = \tilde{p}(x), \quad x \in [\underline{x}_1, \underline{x}]. \quad (\text{IA.48})$$

Theorem 3 implies that $\mathcal{L}^{\bar{x}}p(x) \leq 0$ for any $x < \bar{x}$. Combining it with $\mathcal{L}^{\bar{x}}p(\underline{x}-) = \mathcal{L}^{\bar{x}}p(\underline{x}+) = 0$, we have

$$0 \leq \frac{d}{dx}\mathcal{L}^{\bar{x}}p(x)|_{x=\underline{x}-0} = \frac{d}{dx}\mathcal{L}^{\bar{x}}\tilde{p}(x)|_{x=\underline{x}-0}, \quad (\text{IA.49})$$

where the equality follows from (IA.38) and (IA.48).

Step 3: We prove that $\mathcal{L}^{\bar{x}}\tilde{p}(x) < 0$ for any $x < \underline{x}$.

For any $x \in [0, \bar{x})$, we have

$$\mathbb{E}[Z\tilde{p}(x/Z)] = \int_{x/\bar{x}}^1 z\tilde{p}(x/z)dF(z), \quad (\text{IA.50})$$

and

$$c_x(x; \bar{x}) = r + \lambda F\left(\frac{x}{\bar{x}}\right) + \lambda \left(1 - \frac{\ell}{\bar{x}}\right) \frac{x}{\bar{x}} F'\left(\frac{x}{\bar{x}}\right). \quad (\text{IA.51})$$

Using (IA.47), (IA.50), and $\tilde{p}_x(x) = -1$ for $x \in [0, \underline{x}]$, we obtain, for any $x \in [0, \underline{x}]$,

$$\begin{aligned} \frac{d}{dx}\mathcal{L}^{\bar{x}}\tilde{p}(x) &= -A'(x) + \lambda \left[1 + \int_{x/\bar{x}}^1 \tilde{p}_x(x/z)dF(z) \right] + (r - \mu) \\ &= -[A'(x) - (r - \mu) - \lambda F(x/\bar{x})] + \lambda \int_{x/\bar{x}}^1 (1 + \tilde{p}_x(x/z))dF(z). \end{aligned} \quad (\text{IA.52})$$

Since $(1 - \tau)(1 - \frac{\ell}{\bar{x}})zF'(z) - \tau F(z)$ is strictly increasing in z , we have $(1 - \tau)(1 - \frac{\ell}{\bar{x}})zF'(z) - \tau F(z) > 0$ for any $z \in (0, 1]$. It follows that $(1 - \frac{\ell}{\bar{x}})F'(z) > 0$ for any $z \in (0, 1]$. For any x such that $c(x; \bar{x}) > 1$, we have $\theta(c(x; \bar{x})) = 0$, $A(x) = c(x; \bar{x}) - 1 - \mu x$, and

$$A'(x) - (r - \mu) - \lambda F\left(\frac{x}{\bar{x}}\right) = c_x(x; \bar{x}) - r - \lambda F\left(\frac{x}{\bar{x}}\right) = \lambda \left(1 - \frac{\ell}{\bar{x}}\right) \frac{x}{\bar{x}} F'\left(\frac{x}{\bar{x}}\right) > 0, \quad (\text{IA.53})$$

where the second equality uses (IA.51). Substituting (IA.53) into (IA.52) and using $1 + \tilde{p}_x(x) \leq 0$, we have $\frac{d}{dx} \mathcal{L}^{\bar{x}} \tilde{p}(x) < 0$. Comparing this with (IA.49), we conclude that $c(\underline{x}; \bar{x}) \leq 1$. By (IA.51), $c(x; \bar{x})$ is strictly increasing in $x \in [0, \bar{x})$. Hence, together with $c(\underline{x}; \bar{x}) \leq 1$, this implies $c(x; \bar{x}) < 1$ for any $x \in [0, \underline{x})$. It follows that $\theta(c(x; \bar{x})) = \tau(1 - c(x; \bar{x}))$, $A(x) = \tau + (1 - \tau)c(x; \bar{x}) - 1 - \mu x$ and

$$\begin{aligned} A'(x) - (r - \mu) - \lambda F(x/\bar{x}) &= (1 - \tau)c_x(x; \bar{x}) - r - \lambda F\left(\frac{x}{\bar{x}}\right) \\ &= (1 - \tau)\lambda \left(1 - \frac{\ell}{\bar{x}}\right) \frac{x}{\bar{x}} F'\left(\frac{x}{\bar{x}}\right) - \tau \left[r + \lambda F\left(\frac{x}{\bar{x}}\right)\right], \end{aligned} \quad (\text{IA.54})$$

where the second equality uses (IA.51).

For any $x \in [0, \underline{x}]$, by changing the variable $x/z = u$, we obtain

$$\begin{aligned} \int_{x/\bar{x}}^1 (1 + \tilde{p}_x(x/z)) dF(z) &= \int_x^{\bar{x}} (1 + \tilde{p}_x(u)) \frac{x}{u^2} F'\left(\frac{x}{u}\right) du \\ &= \int_{\underline{x}}^{\bar{x}} (1 + \tilde{p}_x(u)) \frac{x}{u^2} F'\left(\frac{x}{u}\right) du, \end{aligned} \quad (\text{IA.55})$$

where the second equality follows from $\tilde{p}_x(x) = -1$ for any $x \leq \underline{x}$. Since $(1 - \tau)(1 - \frac{\ell}{\bar{x}})zF'(z) - \tau F(z)$ is strictly increasing in z and $F(z)$ is increasing in z , it follows that $zF'(z)$ is strictly increasing in z . Using this fact together with $1 + \tilde{p}_x(x) \leq 0$, we conclude that the expression in (IA.55) is decreasing in x .

Since $(1 - \tau)(1 - \frac{\ell}{\bar{x}})zF'(z) - \tau F(z)$ is strictly increasing in $z \in [0, 1]$, the expression in (IA.54) is strictly increasing in $x \in [0, \bar{x}]$. Combining this with the fact that (IA.55) is decreasing in $x \in [0, \bar{x}]$, it follows from (IA.52) that $\frac{d}{dx} \mathcal{L}^{\bar{x}} \tilde{p}(x)$ is strictly decreasing in $x \in [0, \underline{x}]$. Hence, by (IA.49), $\frac{d}{dx} \mathcal{L}^{\bar{x}} \tilde{p}(x) > 0$ for any $x \in [0, \underline{x})$. Finally, since $\mathcal{L}^{\bar{x}} \tilde{p}(\underline{x}) = \mathcal{L}^{\bar{x}} p(\underline{x}) = 0$, we have $\mathcal{L}^{\bar{x}} \tilde{p}(x) < 0$ for any $x \in [0, \underline{x})$. ■

Proof of Proposition 4. In the case $h_0 = h_1 = 0$, the characterization and uniqueness of equilibrium are proved in Appendix IB.3. Thus, we only need to consider the case $h_0 > 0$ or $h_1 > 0$.

According to the proof of Proposition 3, we have $\bar{x} > \ell$, $p^N(\bar{x}) = p(\bar{x})$, and the firm must issue equity at $\bar{x}$. Then there exists a smallest $\tilde{x} \in [\underline{x}, \bar{x}]$, such that

$$p(\bar{x}) = \max_{x \leq \bar{x}} p(x) - h_0 - (1 + h_1)(\bar{x} - x) = p(\tilde{x}) - h_0 - (1 + h_1)(\bar{x} - \tilde{x}). \quad (\text{IA.56})$$

Since $p(x)$ is the equilibrium equity value and the firm can issue equity to reach any $x \leq \bar{x}$ starting from $\bar{x}$, we have $0 = p(\bar{x}) \geq p(x) - h_0 - (1 + h_1)(\bar{x} - x)$ for any $x \leq \bar{x}$. It follows that

$$p(x) \leq h_0 + (1 + h_1)(\bar{x} - x), \quad x \leq \bar{x}. \quad (\text{IA.57})$$

Since $p(x)$ is the equilibrium equity value and the firm can issue equity to reach $\tilde{x}$ starting from $x \in [\tilde{x}, \bar{x}]$, we have

$$\begin{aligned} p(x) &\geq p(\tilde{x}) - h_0 - (1 + h_1)(x - \tilde{x}) = p(\tilde{x}) - h_0 - (1 + h_1)(\bar{x} - \tilde{x}) - (1 + h_1)(x - \bar{x}) \\ &= p(\bar{x}) + (1 + h_1)(\bar{x} - x) = (1 + h_1)(\bar{x} - x), \quad x \in [\tilde{x}, \bar{x}], \end{aligned} \quad (\text{IA.58})$$

where the second equality uses (IA.56).

The remainder of the proof is divided into four steps.

Step 1: We prove that $\mathcal{L}^{\bar{x}}G(x) < 0$ for any $x \in [\underline{x}, \tilde{x})$, where $G(x) = [p(x) - (1 + h_1)(\bar{x} - x)]\mathbf{1}_{x \leq \bar{x}}$.

Since $h_0 > 0$ or $h_1 > 0$, it is not optimal for the firm to make a payout immediately after issuing equity, implying $\tilde{x} \geq \underline{x}$. Next, we prove that there exists a closed interval containing $\tilde{x}$, such that the firm neither issues equity nor makes a payout in the interior of this interval.

We first consider the case $h_0 > 0$. By the continuity of $p(x)$, there exists $\epsilon > 0$, such that $p(x) \in [p(\tilde{x}) - h_0/4, p(\tilde{x}) + h_0/4]$ for any $x \in [\tilde{x} - \epsilon, \tilde{x} + \epsilon]$. Thus, for any $x \in [\tilde{x} - \epsilon, \tilde{x} + \epsilon]$ and any $m > 0$ such that $x - m \geq \tilde{x} - \epsilon$, we have

$$p(x - m) - h_0 - (1 + h_1)m < p(x). \quad (\text{IA.59})$$

Suppose, toward a contradiction, that there exists a sequence $\{x_n\}_{n \geq 1}$, such that the firm optimally issues equity at x_n , and $\lim_{n \rightarrow \infty} x_n = \tilde{x}$. Since (IA.59) holds whenever $x - m \geq \tilde{x} - \epsilon$, we must have

$$p(x_n) = \max_{x_n - m < \tilde{x} - \epsilon} p(x_n - m) - h_0 - (1 + h_1)m = \max_{x < \tilde{x} - \epsilon} p(x) - h_0 - (1 + h_1)(x_n - x). \quad (\text{IA.60})$$

Then there exists $\tilde{x}^* < \tilde{x} - \epsilon$, independent of x_n , such that

$$p(x_n) = p(\tilde{x}^*) - h_0 - (1 + h_1)(x_n - \tilde{x}^*). \quad (\text{IA.61})$$

Letting $n \rightarrow \infty$ and using the continuity of $p(x)$, we obtain

$$p(\tilde{x}) = p(\tilde{x}^*) - h_0 - (1 + h_1)(\tilde{x} - \tilde{x}^*). \quad (\text{IA.62})$$

Substituting this equality into (IA.56) yields

$$p(\bar{x}) = p(\tilde{x}^*) - 2h_0 - (1 + h_1)(\bar{x} - \tilde{x}^*) < p(\tilde{x}^*) - h_0 - (1 + h_1)(\bar{x} - \tilde{x}^*), \quad (\text{IA.63})$$

which is a contradiction. Thus, there exists $\epsilon_0 > 0$ such that the firm does not issue equity for any $x \in [\tilde{x} - \epsilon_0, \tilde{x} + \epsilon_0]$. Since $\underline{x} \leq \tilde{x}$, the firm neither issues equity nor makes a payout on the interval $x \in (x_1, \tilde{x} + \epsilon_0)$, where $x_1 = \max\{\tilde{x} - \epsilon_0, \underline{x}\}$.

When $h_0 = 0$, we have $h_1 > 0$ and (IA.46) holds. In this case, under the regularity condition, the equity-financing region $\{x \leq \bar{x} : 1 + h_1 + p_x(x) = 0\}$ can be represented by $\cup_{i=1}^n [a_i, b_i]$, where $a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_n \leq b_n = \bar{x}$. Combining (IA.57) and (IA.58), we obtain $p(x) = (1 + h_1)(\bar{x} - x)$ for any $x \in [\tilde{x}, \bar{x}]$. Hence, the firm issues equity for any $x \in [\tilde{x}, \bar{x}]$. Since

$\tilde{x}$ is the smallest point satisfying (IA.56), it lies on the boundary of the equity financing region. Thus, $a_n = \tilde{x}$, and the firm does not issue equity for any $x \in (b_{n-1}, a_n)$. Moreover, since the firm makes payouts for all $x \leq \underline{x}$, issues equity for all $x \in [\tilde{x}, \bar{x}]$, and $h_1 > 0$, we must have $\underline{x} < \tilde{x}$. Therefore, there exists $\epsilon_0 > 0$ such that the firm neither issues equity nor makes a payout on the interval $x \in (\tilde{x} - \epsilon_0, \tilde{x})$.

It follows that the variational inequality (B.6) reduces to the equation $\mathcal{L}^{\bar{x}}p(x) = 0$ for any $x \in \mathbb{I}$, where $\mathbb{I} = (x_1, \tilde{x} + \epsilon_0)$ when $h_0 > 0$, and $\mathbb{I} = (\tilde{x} - \epsilon_0, \tilde{x})$ when $h_0 = 0$ and $h_1 > 0$. On $\mathbb{I}$, the viscosity solution $p(x)$ is a classical solution. When $h_0 > 0$ and $\underline{x} < \tilde{x}$, we have $x_1 < \tilde{x}$. Thus, $\tilde{x}$ is in the interior of $\mathbb{I}$, which implies that $p_x(x)$ and $p_{xx}(x)$ are continuous at $\tilde{x}$. When $h_0 > 0$ and $\underline{x} = \tilde{x}$, the proof of Proposition 3 shows that $p_x(x)$ and $p_{xx}(x)$ are continuous at $x = \underline{x} = \tilde{x}$.

When $h_0 = 0$ and $h_1 > 0$, recall that $\mathcal{L}^{\bar{x}}p(x) = 0$ for any $x \in (\tilde{x} - \epsilon_0, \tilde{x})$ and $p(x) = (1 + h_1)(\bar{x} - x)$ for any $x \in [\tilde{x}, \bar{x}]$. Then we have $p_x(\tilde{x}) = -1 - h_1$. Since $\mathcal{L}^{\bar{x}}p(\tilde{x}-) = 0$, and $\mathcal{L}^{\bar{x}}p(\tilde{x}+) \leq 0$, we have $0 \geq \mathcal{L}^{\bar{x}}p(\tilde{x}+) - \mathcal{L}^{\bar{x}}p(\tilde{x}-) = \frac{\sigma^2 \tilde{x}^2}{2}[p_{xx}(\tilde{x}+) - p_{xx}(\tilde{x}-)]$. Hence, $p_{xx}(\tilde{x}-) \geq p_{xx}(\tilde{x}+) = 0$. On the other hand, since $p_x(x) \geq -1 - h_1$ for $x \in (\tilde{x} - \epsilon_0, \tilde{x})$ and $p_x(\tilde{x}) = -1 - h_1$, it follows that $p_{xx}(\tilde{x}-) \leq 0$. Therefore, $p_{xx}(\tilde{x}-) = 0 = p_{xx}(\tilde{x}+)$, which implies that $p_{xx}(x)$ is continuous at $x = \tilde{x}$.

In summary, in all cases, $p_x(x)$ and $p_{xx}(x)$ are continuous at $x = \tilde{x}$, and $\mathcal{L}^{\bar{x}}p(\tilde{x}) = 0$.

Using (IA.56) and $p(\bar{x}) = 0$, we obtain

$$h_0 = \max_{z \leq \bar{x}} G(z) = G(\tilde{x}). \quad (\text{IA.64})$$

When $h_0 > 0$, (IA.56) implies that $\tilde{x} < \bar{x}$. Recall that $p_x(x)$ and $p_{xx}(x)$ are continuous at $x = \tilde{x}$. It then follows from (IA.64) that $G'(\tilde{x}) = 0$ and $G''(\tilde{x}) \leq 0$. When $h_0 = 0$ and $h_1 > 0$, recall that $p_x(\tilde{x}) = -1 - h_1$ and $p_{xx}(\tilde{x}) = 0$. Thus, $G'(\tilde{x}) = 0$ and $G''(\tilde{x}) = 0$.

Using $G'(\tilde{x}) = 0$ and $G''(\tilde{x}) \leq 0$, we obtain

$$\begin{aligned} \mathcal{L}^{\bar{x}}G(\tilde{x}) &= A(\tilde{x})G'(\tilde{x}) + \frac{\sigma^2 \tilde{x}^2}{2}G''(\tilde{x}) + \lambda \mathbb{E}[ZG(\tilde{x}/Z) - G(\tilde{x})] - (r - \mu)G(\tilde{x}) \\ &\leq \lambda \mathbb{E}[ZG(\tilde{x}/Z) - G(\tilde{x})] - (r - \mu)G(\tilde{x}) \\ &\leq h_0 \left[\lambda \mathbb{E}[Z - 1] - (r - \mu) \right] = h_0[g - r] \leq 0, \end{aligned} \quad (\text{IA.65})$$

where $A(x) = c(x; \bar{x}) + \theta(c(x; \bar{x})) - 1 - \mu x$, the second inequality follows from (IA.64), and the last inequality follows from $r > g$.

Denote $q(x) = (1 + h_1)(\bar{x} - x)^+$. Then $G(x) = p(x) - q(x)$ for any $x \geq 0$. Using $\mathcal{L}^{\bar{x}}G(x) = \mathcal{L}^{\bar{x}}p(x) - \mathcal{L}^{\bar{x}}q(x)$, $\mathcal{L}^{\bar{x}}G(\tilde{x}) \leq 0$ and $\mathcal{L}^{\bar{x}}p(\tilde{x}) = 0$, we have $\mathcal{L}^{\bar{x}}q(\tilde{x}) \geq 0$. In addition, Condition 2 implies that

$$\mathcal{L}^{\bar{x}}q(\underline{x}) = (1 + h_1)\mathcal{L}^{\bar{x}}(\bar{x} - x)^+ \big|_{x=\underline{x}} > 0. \quad (\text{IA.66})$$

For any $x \in [0, \bar{x}]$, using $q_x(x) = -1 - h_1$, we obtain

$$\begin{aligned}\mathcal{L}^{\bar{x}}q(x) &= -(1 + h_1)A(x) + \lambda \left[\int_{x/\bar{x}}^1 zq(x/z)dF(z) - q(x) \right] - (r - \mu)q(x) \\ &= -(1 + h_1) \left[A(x) - \lambda \int_{x/\bar{x}}^1 (z\bar{x} - x)dF(z) + (r - \mu + \lambda)(\bar{x} - x) \right],\end{aligned}\quad (\text{IA.67})$$

where $A(x) = c(x; \bar{x}) + \theta(c(x; \bar{x})) - 1 - \mu x$. Then for $x \in [0, \bar{x}]$, we have

$$\frac{d}{dx}\mathcal{L}^{\bar{x}}q(x) = -(1 + h_1)[A'(x) - (r - \mu) - \lambda F(x/\bar{x})]. \quad (\text{IA.68})$$

By (IA.51), $c(x; \bar{x})$ is strictly increasing in $x \in [0, \bar{x}]$. Let $x_c = \inf\{x \in [0, \bar{x}] : c(x; \bar{x}) \geq 1\}$ with $\inf \emptyset = \bar{x}$. Using (IA.53), we derive from (IA.68) that $\frac{d}{dx}\mathcal{L}^{\bar{x}}q(x) < 0$ for any $x \in (x_c, \bar{x}]$. For any $x \in [0, x_c)$, using (IA.54), (IA.68), and Condition 1, we conclude that $\frac{d}{dx}\mathcal{L}^{\bar{x}}q(x)$ is strictly decreasing in $x \in [0, x_c)$. Thus, $\mathcal{L}^{\bar{x}}q(x)$ is either strictly decreasing on $[0, x_c)$, or first strictly increasing and then strictly decreasing on $[0, x_c)$. Since we have shown that $\frac{d}{dx}\mathcal{L}^{\bar{x}}q(x) < 0$ for any $x \in (x_c, \bar{x}]$, we conclude that $\mathcal{L}^{\bar{x}}q(x)$ is either strictly decreasing on $[0, \bar{x}]$, or first strictly increasing and then strictly decreasing on $[0, \bar{x}]$. Recall that $\mathcal{L}^{\bar{x}}q(\tilde{x}) \geq 0$, and that (IA.66) shows $\mathcal{L}^{\bar{x}}q(\underline{x}) > 0$. Hence, $\mathcal{L}^{\bar{x}}q(x) > 0$ for any $x \in [\underline{x}, \tilde{x})$. Finally, using $\mathcal{L}^{\bar{x}}G(x) = \mathcal{L}^{\bar{x}}p(x) - \mathcal{L}^{\bar{x}}q(x)$ together with $\mathcal{L}^{\bar{x}}p(x) \leq 0$, $\mathcal{L}^{\bar{x}}q(x) > 0$ for any $x \in [\underline{x}, \tilde{x})$, we obtain $\mathcal{L}^{\bar{x}}G(x) < 0$ for any $x \in [\underline{x}, \tilde{x})$.

Step 2: We prove that the firm's equity financing region is $[\tilde{x}, \bar{x}]$ in the case $h_0 = 0$.

When $h_0 = 0$, we have $h_1 > 0$. Moreover, (IA.46) implies that $p_x(x) \geq -1 - h_1$. Hence, $G(x)$ is increasing in x . Using (IA.64) and $h_0 = 0$, we obtain $G(x) \leq 0$ for all $x \geq 0$. Recall that when $h_0 = 0$ and $h_1 > 0$, Step 1 has already shown that the firm issues equity for every $x \in [\tilde{x}, \bar{x}]$, and $\tilde{x} > \underline{x}$. Thus, we only need to show that the firm does not issue equity in the region $x \in (\underline{x}, \tilde{x})$.

Let $\hat{x}$ denote the left endpoint of the lowest boundary of the equity financing region. Since $h_1 > 0$, we have $\hat{x} > \underline{x}$. Suppose, toward a contradiction, that $\hat{x} < \tilde{x}$. Following Step 1, the variational inequality (B.6) reduces to $\mathcal{L}^{\bar{x}}p(x) = 0$ on $(\underline{x}, \hat{x})$, on which the viscosity solution $p(x)$ is a classical solution and satisfies $p_x(\hat{x}) = -1 - h_1$ and $p_{xx}(\hat{x}) = 0$. It follows that $G'(\hat{x}) = 0$ and $G''(\hat{x}) = 0$. Therefore,

$$\begin{aligned}\mathcal{L}^{\bar{x}}G(\hat{x}) &= A(\hat{x})G'(\hat{x}) + \frac{\sigma^2 \hat{x}^2}{2}G''(\hat{x}) + \lambda \mathbb{E}[ZG(\hat{x}/Z) - G(\hat{x})] - (r - \mu)G(\hat{x}) \\ &= \lambda \mathbb{E}[ZG(\hat{x}/Z) - G(\hat{x})] - (r - \mu)G(\hat{x}) \\ &\geq \lambda G(\hat{x})\mathbb{E}[Z - 1] - (r - \mu)G(\hat{x}) = G(\hat{x})[g - r] \geq 0,\end{aligned}\quad (\text{IA.69})$$

where the first inequality uses the fact that $G(x)$ is increasing in x , and the last inequality follows from $G(\hat{x}) \leq 0$ and $g < r$. However, Step 1 shows that $\mathcal{L}^{\bar{x}}G(x) < 0$ for any $x \in [\underline{x}, \tilde{x})$, which gives a contradiction. Thus, $\hat{x} \geq \tilde{x}$, and the firm does not issue equity for any $x \in (\underline{x}, \tilde{x})$.

Step 3: We prove that, when $h_0 > 0$, the firm does not issue equity for any $x \in (\underline{x}, \tilde{x})$.

Recall that $\hat{x}$ denotes the left endpoint of the lowest boundary of the equity financing region. Since we have shown that the firm issues equity at $\bar{x}$, we only need to consider the case $\hat{x} < \bar{x}$. Since $h_0 > 0$, there exists a smallest $\hat{x}_1 < \hat{x}$, such that

$$p(\hat{x}) = \max_{z \leq \hat{x}} p(z) - h_0 - (1 + h_1)(\hat{x} - z) = p(\hat{x}_1) - h_0 - (1 + h_1)(\hat{x} - \hat{x}_1). \quad (\text{IA.70})$$

Moreover, $\hat{x} > \underline{x}$ since $h_0 > 0$ and $[0, \underline{x}]$ is the payout region. Thus, the firm neither issues equity nor makes a payout on the interval $(\underline{x}, \hat{x})$. It follows that the variational inequality (B.6) reduces to the equation $\mathcal{L}^{\bar{x}}p(x) = 0$ for any $x \in (\underline{x}, \hat{x})$, with the smooth-pasting conditions imposed at both $\underline{x}$ and $\hat{x}$. In this region $(\underline{x}, \hat{x})$, the viscosity solution $p(x)$ is a classical solution.

Define a function $\tilde{p}(x)$ as follows:

$$\tilde{p}(x) = 0, \quad x > \bar{x}, \quad (\text{IA.71})$$

$$\tilde{p}(x) = p(\hat{x}) - (1 + h_1)(x - \hat{x}), \quad x \in [\hat{x}, \bar{x}], \quad (\text{IA.72})$$

$$\tilde{p}(x) = p(x), \quad x \leq \hat{x}. \quad (\text{IA.73})$$

Since $p(x)$ is the equilibrium equity value and, starting from any $x \in [\hat{x}, \bar{x}]$, the firm can issue equity to reach $\hat{x}_1$, we have

$$\begin{aligned} p(x) &\geq p(\hat{x}_1) - h_0 - (1 + h_1)(x - \hat{x}_1) \\ &= p(\hat{x}_1) - h_0 - (1 + h_1)(\hat{x} - \hat{x}_1) - (1 + h_1)(x - \hat{x}) \\ &= p(\hat{x}) - (1 + h_1)(x - \hat{x}) = \tilde{p}(x), \end{aligned} \quad (\text{IA.74})$$

where the second equality follows from (IA.70). Setting $x = \bar{x}$ in (IA.74) yields $\tilde{p}(\bar{x}) \leq p(\bar{x}) = 0$. Moreover, since $\tilde{p}_x(x) = q_x(x) = -1 - h_1$ for any $x \in (\hat{x}, \bar{x})$, and $q(\bar{x}) = 0 \geq \tilde{p}(\bar{x})$, it follows that $q(x) \geq \tilde{p}(x)$ for any $x \in [\hat{x}, \bar{x}]$. Since $q(x) = \tilde{p}(x) = 0$ for any $x > \bar{x}$, we conclude that $q(x) \geq \tilde{p}(x)$ for any $x \geq \hat{x}$.

Since $p(x)$ is the equilibrium equity value, for any $x \in [\hat{x}_1, \hat{x} + \epsilon]$, where $\epsilon \in (0, \bar{x} - \hat{x})$, we have

$$\begin{aligned} p(x) &\geq p(\hat{x}_1) - h_0 - (1 + h_1)(x - \hat{x}_1) = p(\hat{x}_1) - h_0 - (1 + h_1)(\hat{x} - \hat{x}_1) - (1 + h_1)(x - \hat{x}) \\ &= p(\hat{x}) - (1 + h_1)(x - \hat{x}), \end{aligned} \quad (\text{IA.75})$$

where the last equality uses (IA.70). Since (IA.75) holds with equality at $x = \hat{x}$, $p(x) - [p(\hat{x}) - (1 + h_1)(x - \hat{x})]$ attains a local minimum at $x = \hat{x}$. This implies that $p_x(\hat{x}) + 1 + h_1 = 0$ and $p_{xx}(\hat{x}-) \geq 0$. Using (IA.74), together with the fact that $\tilde{p}(x) = p(x)$ for $x \leq \hat{x}$ and $x > \bar{x}$, we obtain $\mathbb{E}[Z\tilde{p}(\hat{x}/Z)] \leq \mathbb{E}[Zp(\hat{x}/Z)]$. Combining this with (IA.73), we have $\mathcal{L}^{\bar{x}}\tilde{p}(\hat{x}-) \leq \mathcal{L}^{\bar{x}}p(\hat{x}-) \leq 0$. Since $\tilde{p}_{xx}(\hat{x}-) = p_{xx}(\hat{x}-) \geq 0 = \tilde{p}_{xx}(\hat{x}+)$, and both $\tilde{p}(x)$ and $\tilde{p}_x(x)$ are continuous at $x = \hat{x}$, we have $\mathcal{L}^{\bar{x}}\tilde{p}(\hat{x}+) - \mathcal{L}^{\bar{x}}\tilde{p}(\hat{x}-) = \frac{\sigma^2 \hat{x}^2}{2} [\tilde{p}_{xx}(\hat{x}+) - \tilde{p}_{xx}(\hat{x}-)] \leq 0$. Therefore, $\mathcal{L}^{\bar{x}}\tilde{p}(\hat{x}+) \leq \mathcal{L}^{\bar{x}}\tilde{p}(\hat{x}-) \leq 0$.

Let $\bar{x}_1 = p(\hat{x}) / (1 + h_1) + \hat{x}$. Then, by (IA.72), we have $\tilde{p}(x) = (1 + h_1)(\bar{x}_1 - x)$ for any $x \in [\hat{x}, \bar{x}]$. Since we have shown that $\tilde{p}(\bar{x}) \leq 0$, it follows that $\bar{x}_1 \leq \bar{x}$. For any $x \in [\hat{x}, \bar{x})$, using

$\tilde{p}_x(x) = -1 - h_1$, we obtain

$$\begin{aligned}\mathcal{L}^{\bar{x}}\tilde{p}(x) &= -(1 + h_1)A(x) + \lambda \left[\int_{x/\bar{x}}^1 z\tilde{p}(x/z)dF(z) - \tilde{p}(x) \right] - (r - \mu)\tilde{p}(x) \\ &= -(1 + h_1) \left[A(x) - \lambda \int_{x/\bar{x}}^1 (z\bar{x}_1 - x)dF(z) + (r - \mu + \lambda)(\bar{x}_1 - x) \right].\end{aligned}\quad (\text{IA.76})$$

Note that (IA.76) is linear in $\bar{x}_1$ with coefficient

$$\begin{aligned}& -(1 + h_1) \left[-\lambda \int_{x/\bar{x}}^1 zdF(z) + (r - \mu + \lambda) \right] \\ & \leq -(1 + h_1) \left[-\lambda \int_0^1 zdF(z) + (r - \mu + \lambda) \right] = -(1 + h_1)(r - g) < 0.\end{aligned}\quad (\text{IA.77})$$

Comparing (IA.67) with (IA.76), and using $\bar{x} \geq \bar{x}_1$, we conclude that $\mathcal{L}^{\bar{x}}q(x) \leq \mathcal{L}^{\bar{x}}\tilde{p}(x)$ for any $x \in [\hat{x}, \bar{x}]$. Since we have shown that $\mathcal{L}^{\bar{x}}\tilde{p}(\hat{x}+) \leq 0$, it follows that $\mathcal{L}^{\bar{x}}q(\hat{x}) \leq 0$.

Since $\hat{x} > \underline{x}$ and Step 1 shows that $\mathcal{L}^{\bar{x}}q(x) > 0$ for any $x \in [\underline{x}, \tilde{x}]$, it follows that $\hat{x} \geq \tilde{x}$. Since $\hat{x}$ is the left endpoint of the lowest boundary of the equity financing region, the firm does not issue equity for any $x \in (\underline{x}, \tilde{x})$.

Step 4: We prove that, when $h_0 > 0$, the firm's equity financing region is given by $[\hat{x}, \bar{x}]$, and that $\hat{x} > \tilde{x}$.

Since we have shown that the firm issues equity at $\bar{x}$, it remains only to consider the case $\hat{x} < \bar{x}$. Combining (IA.56) and (IA.72), we obtain

$$\tilde{p}(\bar{x}) - p(\bar{x}) = p(\hat{x}) - [p(\tilde{x}) - h_0 - (1 + h_1)(\hat{x} - \tilde{x})]. \quad (\text{IA.78})$$

Since $p(x)$ is the equilibrium equity value, and since Step 3 shows that $\hat{x} \geq \tilde{x}$, it follows from (IA.78) that $\tilde{p}(\bar{x}) \geq p(\bar{x})$. Combining this result with (IA.74), we obtain $\tilde{p}(\bar{x}) = p(\bar{x}) = 0$.

Using (IA.70) and $G(x) = [p(x) - (1 + h_1)(\bar{x} - x)]\mathbf{1}_{x \leq \bar{x}}$, we obtain

$$\max_{z \leq \hat{x}} G(z) = G(\hat{x}_1) = h_0 + p(\hat{x}) - (1 + h_1)(\bar{x} - \hat{x}) = h_0 + \tilde{p}(\bar{x}) = h_0. \quad (\text{IA.79})$$

Using $\hat{x} \geq \tilde{x}$, and recalling that $\tilde{x}$ is the smallest point satisfying (IA.64), while $\hat{x}_1$ is the smallest point satisfying (IA.79), we have $\tilde{x} = \hat{x}_1$. Since $h_0 > 0$, it follows that $\tilde{x} = \hat{x}_1 < \hat{x}$. Using (IA.65), together with $r > g$ and $h_0 > 0$, we have $\mathcal{L}^{\bar{x}}G(\tilde{x}) < 0$. Combining this with $\mathcal{L}^{\bar{x}}G(x) = \mathcal{L}^{\bar{x}}p(x) - \mathcal{L}^{\bar{x}}q(x)$ and recalling from Step 1 that $\mathcal{L}^{\bar{x}}p(\tilde{x}) = 0$, we obtain $\mathcal{L}^{\bar{x}}q(\tilde{x}) > 0$.

Since Step 1 shows that $\mathcal{L}^{\bar{x}}q(x)$ is either strictly decreasing on $[0, \bar{x})$, or first strictly increasing and then strictly decreasing on $[0, \bar{x})$, and since $\tilde{x} < \hat{x}$, $\mathcal{L}^{\bar{x}}q(\tilde{x}) > 0$ and $\mathcal{L}^{\bar{x}}q(\hat{x}) \leq 0$, we conclude that $\mathcal{L}^{\bar{x}}q(x)$ is strictly decreasing on $[\hat{x}, \bar{x}]$. Hence, $\mathcal{L}^{\bar{x}}q(x) < 0$ for any $x \in (\hat{x}, \bar{x}]$.

Since $\tilde{p}(\bar{x}) = 0$, we derive from (IA.71)-(IA.72) that $\tilde{p}(x) = (1 + h_1)(\bar{x} - x)^+ = q(x)$ for any $x \geq \hat{x}$. Thus, $\mathcal{L}^{\bar{x}}\tilde{p}(x) < 0$ for any $x \in (\hat{x}, \bar{x}]$.

Using (IA.70), we have $\max_{z \leq \hat{x}} p(z) + (1 + h_1)z = p(\hat{x}_1) + (1 + h_1)\hat{x}_1$. Moreover, by (IA.72), $\tilde{p}(x) + (1 + h_1)x = p(\hat{x}) + (1 + h_1)\hat{x}$ for any $x \in [\hat{x}, \bar{x}]$. Thus, for any $x \in [\hat{x}, \bar{x}]$,

$$\begin{aligned} \max_{z \leq x} \tilde{p}(z) - h_0 - (1 + h_1)(x - z) &= \max_{z \leq \hat{x}} p(z) - h_0 - (1 + h_1)(x - z) \\ &= p(\hat{x}) - (1 + h_1)(x - \hat{x}) = \tilde{p}(x), \end{aligned} \quad (\text{IA.80})$$

where the second equality follows from (IA.70) and the last equality follows from (IA.72). Furthermore, by (IA.72), we have $1 + \tilde{p}_x(x) = -h_1 \leq 0$ for any $x \in [\hat{x}, \bar{x}]$. In sum, we have shown that $\tilde{p}(x) = 0$ for $x \geq \bar{x}$, that (IA.80) holds, and that $1 + \tilde{p}_x(x) \leq 0$, $\mathcal{L}^{\bar{x}}\tilde{p}(x) < 0$ for any $x \in (\hat{x}, \bar{x}]$. Hence, $\tilde{p}(x)$ satisfies the variational inequality (B.6) on $(\hat{x}, \bar{x}]$. Since p and $\tilde{p}$ are two continuous bounded viscosity solutions of (B.6) on $(\hat{x}, \bar{x}]$, and since they coincide on $[0, \hat{x}] \cup [\bar{x}, \infty)$, uniqueness of the solution to the variational inequality implies that $\tilde{p}(x) = p(x)$ for all $x \in [\hat{x}, \bar{x}]$. Hence, according to (IA.80), the firm issues equity for all $x \in [\hat{x}, \bar{x}]$. Moreover, $\hat{x} > \tilde{x} \geq \underline{x}$. ■

Proof of Theorem 4. By Lemma 1 and Propositions 3–4, the default, payout, and equity-issuance regions are $(\bar{x}, \infty)$, $[0, \underline{x})$, and $[\hat{x}, \bar{x}]$ with $\underline{x} < \hat{x} \leq \bar{x}$, and $p(x)$ is given by (12) and (23) in the latter two regions. On the remaining interval $[\underline{x}, \hat{x})$ the firm neither pays out, issues equity, nor defaults, so the variational inequality (B.6) of Theorem 3 reduces to $\mathcal{L}^{\bar{x}}p(x) = 0$, which is (21). This proves Theorem 4. ■

Proof of Proposition 5. Fix a state $x_{t-} < \bar{x}$ at time $t-$. We have

$$\int_{s \in [t, T^{\mathcal{D}})} e^{-r(s-t)} (rX_{s-} ds - dX_s) = - \int_{s \in [t, T^{\mathcal{D}})} d[e^{-r(s-t)} X_s] = X_{t-} - e^{-r(T^{\mathcal{D}}-t)} X_{T^{\mathcal{D}}-}. \quad (\text{IA.81})$$

Substituting the contractual interest-payment rule $C_{s-} = [r + \eta(x_{s-}; \bar{x})]X_{s-}$ and (IA.81) into (C.1) yields

$$\begin{aligned} D_{t-}(S) &= X_{t-} + \mathbb{E}_{t-} \left[\int_t^{T^{\mathcal{D}}} e^{-r(s-t)} \eta(x_{s-}; \bar{x}) X_{s-} ds - e^{-r(T^{\mathcal{D}}-t)} (X_{T^{\mathcal{D}}-} - \min\{X_{T^{\mathcal{D}}-}, L_{T^{\mathcal{D}}}\}) \right] \\ &= X_{t-} + \mathbb{E}_{t-} \left[\int_t^{T^{\mathcal{D}}} e^{-r(s-t)} \eta(x_{s-}; \bar{x}) X_{s-} ds - e^{-r(T^{\mathcal{D}}-t)} (X_{T^{\mathcal{D}}-} - L_{T^{\mathcal{D}}})^+ \right], \end{aligned} \quad (\text{IA.82})$$

where the second equality uses $X - \min\{X, L\} = (X - L)^+$.

Under S^* , default occurs at time s if and only if a jump arrives with recovery fraction Z_s satisfying $x_{s-}/Z_s > \bar{x}$, where $x_{s-} = X_{s-}/Y_{s-}$, the compensator formula gives

$$\begin{aligned} &\mathbb{E}_{t-} \left[e^{-r(T^{\mathcal{D}}-t)} (X_{T^{\mathcal{D}}-} - L_{T^{\mathcal{D}}})^+ \right] \\ &= \mathbb{E}_{t-} \left[\int_t^\infty e^{-r(s-t)} \mathbf{1}_{\{s \leq T^{\mathcal{D}}\}} \int_{z < x_{s-}/\bar{x}} (X_{s-} - L_s(z))^+ dF(z) \lambda ds \right], \end{aligned} \quad (\text{IA.83})$$

where $L_s(z) = \ell z Y_{s-}$.

Using (A.6), the contractual spread at every solvent state $x_{s-} < \bar{x}$ satisfies

$$\eta(x_{s-}; \bar{x})X_{s-} = \lambda \mathbb{E}_{s-} [(X_{s-} - L_s(Z_s))^+ \mathbf{1}_{\{x_{s-}/Z_s > \bar{x}\}}] = \lambda \int_{z < x_{s-}/\bar{x}} (X_{s-} - L_s(z))^+ dF(z). \quad (\text{IA.84})$$

Substituting (IA.84) into (IA.83), we obtain

$$\begin{aligned} & \mathbb{E}_{t-} \left[e^{-r(T^\mathcal{D}-t)} (X_{T^\mathcal{D}-} - L_{T^\mathcal{D}})^+ \right] \\ &= \mathbb{E}_{t-} \left[\int_t^\infty e^{-r(s-t)} \mathbf{1}_{\{s \leq T^\mathcal{D}\}} \eta(x_{s-}; \bar{x}) X_{s-} ds \right] = \mathbb{E}_{t-} \left[\int_t^{T^\mathcal{D}} e^{-r(s-t)} \eta(x_{s-}; \bar{x}) X_{s-} ds \right]. \end{aligned} \quad (\text{IA.85})$$

Substituting (IA.85) into (IA.82), we obtain $D_{t-}(S^*) = X_{t-}$ for all $t < T^\mathcal{D}$. In particular, $D_0(S^*) = X_0$, so the lender breaks even at $t = 0$ and (C.2) holds with equality. Thus, the lender has no incentive to renegotiate the long-term contract at any $t < T^\mathcal{D}$.

Fix the contractual spread schedule $\eta(x; \bar{x})$ in (A.7) and the interest-payment rule $C_{t-} = [r + \eta(x_{t-}; \bar{x})]X_{t-}$. For any feasible Markov strategy S , the long-term agreement induces the same state dynamics and equity payoff as the firm's best-response problem (A.3) in the baseline model when creditors' belief is given by S^* . Since S^* is the firm's best response to creditors' belief S^* in the baseline model, it is also optimal under the long-term agreement. This proves that the firm's optimal policy under the long-term contract coincides with S^* . ■

IB Solution for the Costless Equity Issue Case: $h_0 = h_1 = 0$

First, as in our baseline model of Section 2, given shareholders' limited liability, there exists a cutoff level $\bar{x}$ above which the firm defaults so that equity value is zero: $p(x) = 0$ for $x > \bar{x}$. Second, because $h_0 = h_1 = 0$, the payout region, the earnings-retention and debt-rollover region, and the equity issuance region are all combined into one region where $p'(x) = -1$ and $p''(x) = 0$ for $x \leq \bar{x}$. Substituting these two conditions into the integro-differential equation (21) for $p(x)$ at $x = \tilde{x}$, we obtain:

$$(r - \mu)p(\tilde{x}) = -[c(\tilde{x}; \bar{x}) + \theta(c(\tilde{x}; \bar{x})) - 1 - \mu\tilde{x}] + \lambda \mathbb{E}[Zp(\tilde{x}/Z) - p(\tilde{x})], \quad (\text{IB.1})$$

where $\tilde{x} \in [0, \bar{x}]$ denotes the target debt-EBIT ratio and $c(x; \bar{x}) = (r + \eta(x; \bar{x}))x$. Equation (IB.1) shows that cash-flow volatility has no impact on $p(x)$. This is because firms do not value financial flexibility in the absence of equity issuance costs.

Using $c(x; \bar{x}) = (r + \eta(x; \bar{x}))x$, (A.7) for credit spread $\eta(x; \bar{x})$, and the decomposition $v(x) = p(x) + x$, we can rewrite (IB.1) and obtain the following equation for $v(x)$ at $x = \tilde{x}$:

$$\begin{aligned} (r - \mu)v(\tilde{x}) &= (r - \mu)\tilde{x} - [c(\tilde{x}; \bar{x}) + \theta(c(\tilde{x}; \bar{x})) - 1 - \mu\tilde{x}] + \lambda \mathbb{E}[Zp(\tilde{x}/Z) - p(\tilde{x})] \\ &= 1 - \theta(c(\tilde{x}; \bar{x})) - \eta(\tilde{x}; \bar{x})\tilde{x} + \lambda \mathbb{E}[Zp(\tilde{x}/Z) - p(\tilde{x})] \\ &= 1 - \theta(c(\tilde{x}; \bar{x})) + \lambda \mathbb{E}[Zv(\tilde{x}/Z) - v(\tilde{x})]. \end{aligned} \quad (\text{IB.2})$$

Since the firm defaults if and only if x exceeds $\bar{x}$, we have $v(x) = \ell$ for $x > \bar{x}$. Also note that $v(x) = v(\bar{x}) = \bar{x}$ for all $x \leq \bar{x}$ because equity issuance is costless. Using these results and $g = \mu - \lambda[1 - \mathbb{E}(Z)]$ to rewrite (IB.2), we obtain:

$$v(x) = \bar{x} = \frac{1}{r - g} \left[1 - \underbrace{\theta(c(\tilde{x}; \bar{x}))}_{\text{taxes}} - \underbrace{\lambda(\bar{x} - \ell) \left(\int_0^{\tilde{x}/\bar{x}} z dF(z) \right)}_{\text{expected firm loss given default}} \right], \quad x < \bar{x}, \quad (\text{IB.3})$$

which defines an implicit function for $\bar{x} = \bar{x}(\tilde{x})$; see Proposition 6 in Internet Appendix IB.2 for further details. Finally, equityholders choose the optimal target debt-EBIT ratio $\tilde{x} = x^*$ to maximize $v(x) = \bar{x}$ given in (IB.3). The firm value associated with this optimal policy is $v(x) = \bar{x}(x^*)$ and the equity value is $p(x) = \bar{x}(x^*) - x$ for any $x < \bar{x}(x^*)$.

Next, we provide closed-form solutions for two special cases of our costless equity issuance benchmark: (1.) the no-jump case ($\lambda = 0$) and (2.) the jump case with no recovery ($Z = 0$).

For the costless equity issuance no-jump ($\lambda = 0$) case, shareholders set the target of x at

$$x^* = \frac{1}{r - \mu \left(\frac{1}{1-\tau} \mathbf{1}_{\mu \leq 0} + \mathbf{1}_{\mu > 0} \right)}, \quad (\text{IB.4})$$

at all times and never default. In effect, shareholders sell the firm to creditors by setting a 100% market leverage: $v(x^*) = x^*$ and $p(x^*) = 0$. The firm avoids inefficient liquidation by issuing equity at no cost.

For the costless equity issuance jump case with no recovery ($Z = 0$), as in Hu, Varas, and Ying (2026), the firm defaults when a jump arrives. The equilibrium credit spread is $\eta = \lambda$. Shareholders effectively sell the firm to creditors by setting the target of x at

$$x^* = \frac{1}{r + \lambda - \mu \left(\frac{1}{1-\tau} \mathbf{1}_{\mu \leq 0} + \mathbf{1}_{\mu > 0} \right)}. \quad (\text{IB.5})$$

For both special cases, debt has tax benefits and no cost. Therefore, the firm chooses 100% market leverage by delaying equity issuance as much as possible and by setting the optimal target x^* to the equilibrium debt capacity $\bar{x}$. If $\lambda = 0$, the firm has no default risk as diffusion shocks cannot take a firm's x to the right of its equilibrium debt capacity $\bar{x}$. If $\lambda > 0$ but $Z = 0$, default is purely exogenous. In this case, the firm also defers its equity issuance as much as it can, i.e., until $x = \bar{x}$.

Finally, we provide proofs for the case where $\lambda > 0$ and $F'(z) > 0$, $z \in [0, 1]$. In Appendix IB.1, we prove the existence of an MPE and the best equilibrium. In Appendix IB.2, we prove that the best equilibrium can be solved by choosing the optimal target debt-EBIT ratio $\tilde{x}$ to maximize $\bar{x}$ given in (IB.3). In Appendix IB.3, we prove the uniqueness of the equilibrium strategy.

IB.1 Existence of an MPE and the Best Equilibrium

In this section, we prove the existence of a Markov perfect equilibrium, and moreover, prove that there exists a best equilibrium with the highest firm value among all equilibria in the set $\mathcal{S}_x$.

Lemma 3. Under Assumption 1, given creditors' belief $\tilde{S}$ with a default region $\{(Y, X) : X/Y > \bar{x}\}$ for some constant $\bar{x} \geq \ell$, the optimal equity value is given by

$$P(Y, X) = (\mathcal{A}(\bar{x})Y - X)^+, \quad (\text{IB.6})$$

and the firm's optimal strategy is to set X_t/Y_t as a constant for any $t < T^D$, where $\mathcal{A}(\bar{x}) = P(1, 0) > \ell$. Additionally, we have $\mathcal{A}(\bar{x}) < \bar{x}$ if $\bar{x} > \frac{1}{r-g}$.

Lemma 4. Under Assumption 1, given creditors' belief $\tilde{S}$ with a default region $\{(Y, X) : X/Y > \bar{x}\}$ for some constant $\bar{x} \geq \ell$, setting the target of x_t at $\tilde{x}$ is a feasible strategy if and only if $\tilde{x} \in \mathbb{U}(\bar{x})$, where

$$\mathbb{U}(\bar{x}) = \{\tilde{x} \in [0, \bar{x}] : 1 + \mu\tilde{x} - \theta(c(\tilde{x}; \bar{x})) - c(\tilde{x}; \bar{x}) \geq 0\}, \quad (\text{IB.7})$$

and $c(x; \bar{x}) = rx + \lambda \int_0^{x/\bar{x}} (x - \ell z) dF(z)$. We have $\mathbb{U}(\bar{x}) \subseteq \mathbb{U}(\bar{x} + \Delta x)$ for any $\Delta x > 0$. Furthermore, given the above pair $(\tilde{x}, \bar{x})$, the corresponding firm value is the unique solution $\tilde{v} \geq \tilde{x}$ to the following equation

$$(r - g)\tilde{v} - \lambda \int_0^{\tilde{x}/\tilde{v}} (\tilde{x} - z\tilde{v}) dF(z) = r\tilde{x} + 1 - \theta(c(\tilde{x}; \bar{x})) - c(\tilde{x}; \bar{x}). \quad (\text{IB.8})$$

The solution $\tilde{v}(\tilde{x}, \bar{x})$ to (IB.8) is nondecreasing in $\bar{x}$, and $\tilde{v}(\tilde{x}, \bar{x}) < \bar{x}$ for any $\bar{x} > \frac{1}{r-g}$, $\tilde{x} \in \mathbb{U}(\bar{x})$, and

$$\mathcal{A}(\bar{x}) = \max_{\tilde{x} \in \mathbb{U}(\bar{x})} \tilde{v}(\tilde{x}, \bar{x}), \quad \bar{x} \geq \ell, \quad (\text{IB.9})$$

where $\mathcal{A}(\bar{x})$ is given in Lemma 3. Moreover, $\mathcal{A}(\bar{x})$ is increasing in $\bar{x} \geq \ell$.

The existence of equilibrium boils down to finding the fixed point of $\mathcal{A}(\cdot)$. The next theorem proves the existence of an MPE.

Theorem 5. Suppose Assumption 1 holds. There exists $\bar{x}^* > \ell$ and $\tilde{x}^* \in \mathbb{U}(\bar{x}^*)$, such that $\bar{x}^* = \mathcal{A}(\bar{x}^*) = \tilde{v}(\tilde{x}^*, \bar{x}^*)$, and the strategy S with a default region $\{(Y, X) : X/Y > \bar{x}^*\}$ and target $\tilde{x}^*$ for X_t/Y_t is an MPE. Furthermore, there exists a best equilibrium which yields the highest firm value in the set $\mathcal{S}_x$.

IB.2 Solving for the best MPE

In this section, we show that the best MPE can be solved by choosing the optimal recapitalization target debt-EBIT ratio to maximize firm value given by (IB.3).

Define the set of feasible recapitalization target debt-EBIT ratios as follows:

$$\tilde{\mathbb{U}} = \{\tilde{x} \geq 0 : \exists \bar{x} \geq \tilde{x}, \text{ s.t. (IB.3) holds } \}. \quad (\text{IB.10})$$

Proposition 6. *Suppose Assumption 1 holds. We have*

$$\tilde{\mathcal{U}} = [0, \tilde{x}_{max}], \quad (\text{IB.11})$$

for some $\tilde{x}_{max} = \max_{z \in [0,1]} z f_0(z) > 0$. For any $\tilde{x} \in \tilde{\mathcal{U}}$ and $\bar{x} \geq \tilde{x}$ satisfying (IB.3), we have

$$\tilde{x} = z f_0(z), \quad \bar{x} = f_0(z), \quad (\text{IB.12})$$

where $z = \tilde{x}/\bar{x} \in [0, 1]$, and $f_0(z)$ is defined by

$$f_0(z) = \begin{cases} f_1(z), & \text{if } \mu \leq 0, \quad z \in [0, 1], \\ f_1(z), & \text{if } \mu > 0, \quad z \in [0, z_0], \\ f_2(z), & \text{if } \mu > 0, \quad z \in (z_0, 1]. \end{cases} \quad (\text{IB.13})$$

In (IB.13), $f_1(z)$ and $f_2(z)$ are given by

$$f_1(z) = \frac{(1 - \tau) \left[1 + \lambda \ell \int_0^z Z dF(Z) \right]}{r - g - \tau z [r + \lambda F(z)] + \lambda \int_0^z Z dF(Z)} \quad \text{and} \quad f_2(z) = \frac{1 + \lambda \ell \int_0^z Z dF(Z)}{r - g + \lambda \int_0^z Z dF(Z)},$$

respectively. When $\mu > 0$, $z_0 \in (0, 1)$ is the unique solution to $r - g = r z_0 + \lambda \int_0^{z_0} (z_0 - u) dF(u)$, and satisfies $f_1(z_0) = f_2(z_0)$.

Theorem 6. *Under Assumption 1, the best equilibrium in Theorem 5 is characterized by the default boundary $\bar{x}_* > \ell$ and the target $\tilde{x}_*$, where*

$$\bar{x}_* = \max_{z \in [0,1]} f_0(z) \quad \text{and} \quad \tilde{x}_* = \bar{x}_* z_*. \quad (\text{IB.14})$$

Here, $f_0(z)$ is given in Proposition 6 and z_* is any maximum point of $f_0(z)$ in the region $[0, 1]$.

IB.3 Uniqueness of the Equilibrium Strategy

In this section, we prove the uniqueness of the equilibrium under the following assumption⁵⁰.

Assumption 3. $zF'(z)/F(z)$ is nondecreasing in $z \in [0, 1]$.

Proposition 7. *Under Assumptions 1 and 3,*

- (i) *there exists a unique $z_* \in [0, 1]$, such that $f_0(z_*) = \max_{z \in [0,1]} f_0(z)$ in (IB.14);*
- (ii) *the unique $\bar{x}$ satisfying $\mathcal{A}(\bar{x}) = \bar{x}$ is given by $\bar{x} = \bar{x}_*$ defined in (IB.14), where $\mathcal{A}(\bar{x})$ is given in (IB.9).*

As a result, there exists a unique equilibrium in the set $\mathcal{S}_x$.

Proposition 7-(i) implies that there exists a unique target debt-EBIT ratio $\tilde{x}_* = \bar{x}_* z_*$ achieving the best equilibrium value $\bar{x}_*$; Proposition 7-(ii) proves that $\bar{x}_*$ is the unique equilibrium firm value.

⁵⁰ For example, Assumption 3 holds for the power law $F(z) = z^\beta$, where $\beta > 0$.

IB.4 Proofs for the Costless Equity Issuance Case

Proof of Lemma 3. Substituting (9) into (10), we obtain the following expression for the equity value

$$P(Y_t, X_{t-}) = \mathbb{E}_t \left[\int_t^{T^\mathcal{D}} e^{-r(s-t)} \left[(Y_s - (r + \eta_s)X_s - \Theta_s) ds + dX_s \right] \right], \quad (\text{IB.15})$$

Using $d(e^{-rs}X_s) = e^{-rs}[-rX_s ds + dX_s]$ and the fact that the firm neither makes a payout nor issues equity when it defaults, we have

$$\mathbb{E}_t \left[\int_t^{T^\mathcal{D}} e^{-r(s-t)} \left[-rX_s ds + dX_s \right] \right] = \mathbb{E}_t \left[e^{-r(T^\mathcal{D}-t)} X_{T^\mathcal{D}-} - X_{t-} \right]. \quad (\text{IB.16})$$

Substituting the above into (IB.15), we obtain

$$P(Y_t, X_{t-}) = \mathbb{E}_t \left[\int_t^{T^\mathcal{D}} e^{-r(s-t)} \left[(Y_s - \eta_s X_s - \Theta_s) ds \right] + e^{-r(T^\mathcal{D}-t)} X_{T^\mathcal{D}-} \right] - X_{t-}. \quad (\text{IB.17})$$

It follows that equity value $P(Y_t, X_{t-}) = P(Y_t, 0) - X_{t-}$ for any $X_{t-} < P(Y_t, 0)$ and $P(Y_t, X_{t-}) = 0$ for any $X_{t-} \geq P(Y_t, 0)$. Thus, firm value $V(Y_t, X_{t-}) = P(Y_t, X_{t-}) + X_{t-} = P(Y_t, 0)$ for any $X_{t-} < P(Y_t, 0)$.

Denote $X_s/Y_s = x_s$. We have

$$P(Y_t, 0) = \mathbb{E}_t \left[\int_t^{T^\mathcal{D}} e^{-r(s-t)} \left[(1 - \eta(x_s; \bar{x})x_s - \theta(c_s)) Y_s ds \right] + e^{-r(T^\mathcal{D}-t)} x_{T^\mathcal{D}-} Y_{T^\mathcal{D}-} \right]. \quad (\text{IB.18})$$

where $\eta(x; \bar{x})$ is given in (A.7), $\theta(c)$ is given in (5) and $c_s = rx_s + \eta(x_s; \bar{x})x_s$.

Let $G_T = \Pi_{s \in \{s \in [t, T]: d\mathcal{J}_s = 1\}} Z_s$ and $\tilde{G}_T = e^{-\frac{1}{2}\sigma^2(T-t) + \sigma(\mathcal{B}_T - \mathcal{B}_t)}$, so that $Y_T = Y_t e^{\mu(T-t)} G_T \tilde{G}_T$. We define the new probability measure $\tilde{\mathbb{P}}(D) = \mathbb{E}_t[\mathbf{1}_D \tilde{G}_T]$ for any event $D \in \mathcal{F}_T$, $T \geq t$, and denote by $\tilde{\mathbb{E}}$ the expectation under the new probability measure $\tilde{\mathbb{P}}$. Then we can derive from (IB.18) that

$$P(Y_t, 0) = Y_t \tilde{\mathbb{E}}_t \left[\int_t^{T^\mathcal{D}} e^{-(r-\mu)(s-t)} \left[(1 - \eta(x_s; \bar{x})x_s - \theta(c_s)) G_s ds \right] + e^{-(r-\mu)(T^\mathcal{D}-t)} x_{T^\mathcal{D}-} G_{T^\mathcal{D}-} \right]. \quad (\text{IB.19})$$

By the homogeneity of (IB.19), we conclude that $P(Y_t, 0) = Y_t P(1, 0)$. Moreover, since $P(1, 0)$ is the equity value under the optimal strategy, $\{x_s\}_{s < T^\mathcal{D}}$ is the optimal solution to the following problem:

$$\begin{aligned} & P(1, 0) \\ &= \max_{\{\tilde{x}_s\}_{s < T^\mathcal{D}}} \tilde{\mathbb{E}}_t \left[\int_t^{T^\mathcal{D}} e^{-(r-\mu)(s-t)} \left[(1 - \eta(\tilde{x}_s; \bar{x})\tilde{x}_s - \theta(c_s)) G_s ds \right] + e^{-(r-\mu)(T^\mathcal{D}-t)} \tilde{x}_{T^\mathcal{D}-} G_{T^\mathcal{D}-} \right]. \end{aligned} \quad (\text{IB.20})$$

Since $\mathcal{J}_s$ is a Poisson process with a constant arrival rate, it is optimal to keep x_s constant for all $s < T^D$. Let $\mathcal{A}(\bar{x}) = P(1, 0)$. We have $P(Y, X) = \mathcal{A}(\bar{x})Y - X$ for $X < \mathcal{A}(\bar{x})Y$, and $P(Y, X) = 0$ for $X \geq \mathcal{A}(\bar{x})Y$. Then (IB.6) holds.

Because $X_s = 0$ for any $s \geq t$ and $T^D = \infty$ is a suboptimal strategy starting from $X_t = 0$, we have

$$P(Y_t, 0) \geq \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} (1 - \tau) Y_s ds \right] = \frac{1 - \tau}{r - g} Y_t > \ell Y_t. \quad (\text{IB.21})$$

It follows that $\mathcal{A}(\bar{x}) > \ell$.

In what follows below, if a jump arrives at time s ($d\mathcal{J}_s = 1$), we denote the jump size as Z_s , so that the firm's EBIT decreases from Y_{s-} to $Y_s = Z_s Y_{s-}$. Let $T_1 = \inf\{s \geq t : d\mathcal{J}_s = 1\}$. For the sake of contradiction, assume that $\mathcal{A}(\bar{x}) \geq \bar{x}$ when $\bar{x} > \frac{1}{r-g}$. When $x_t < \mathcal{A}(\bar{x})$, we have $x_s < \mathcal{A}(\bar{x})$ for any $s \in (t, T_1)$ and the firm won't default before time T_1 . Thus, for any $X_{t-} < P(Y_t, 0)$,

$$\begin{aligned} P(Y_t, X_{t-}) &= \mathbb{E}_t \left[\int_t^{T_1-} e^{-r(s-t)} \left[(Y_s - (r + \eta_s) X_s - \Theta_s) ds + dX_s \right] \right] \\ &\quad + \mathbb{E}_t \left[e^{-r(T_1-t)} P(Z_{T_1} Y_{T_1-}, X_{T_1-}) \right] \\ &= \mathbb{E}_t \left[\int_t^{T_1-} e^{-r(s-t)} (Y_s - \eta_s X_s - \Theta_s) ds \right] - X_{t-} \\ &\quad + \mathbb{E}_t \left[e^{-r(T_1-t)} (P(Z_{T_1} Y_{T_1-}, X_{T_1-}) + X_{T_1-}) \right], \end{aligned} \quad (\text{IB.22})$$

where the second equality uses $\mathbb{E}_t \left[\int_t^{T_1-} e^{-r(s-t)} \left[-r X_s ds + dX_s \right] \right] = \mathbb{E}_t \left[e^{-r(T_1-t)} X_{T_1-} - X_{t-} \right]$. Adding X_{t-} to both sides of (IB.22) and using $P(Y_t, X_{t-}) + X_{t-} = P(Y_t, 0)$ for $X_{t-} < P(Y_t, 0)$, we obtain

$$\begin{aligned} P(Y_t, 0) &= \mathbb{E}_t \left[\int_t^{T_1-} e^{-r(s-t)} (Y_s - \eta_s X_s - \Theta_s) ds \right] + \mathbb{E}_t \left[e^{-r(T_1-t)} (P(Z_{T_1} Y_{T_1-}, X_{T_1-}) + X_{T_1-}) \right] \\ &\leq \mathbb{E}_t \left[\int_t^{T_1-} e^{-r(s-t)} (Y_s - \eta_s X_s) ds \right] \\ &\quad + \mathbb{E}_t \left[e^{-r(T_1-t)} (P(Z_{T_1} Y_{T_1-}, X_{T_1-}) + X_{T_1-}) \mathbf{1}_{P(Z_{T_1} Y_{T_1-}, 0) > X_{T_1-}} \right] \\ &\quad + \mathbb{E}_t \left[e^{-r(T_1-t)} (P(Z_{T_1} Y_{T_1-}, X_{T_1-}) + X_{T_1-}) \mathbf{1}_{P(Z_{T_1} Y_{T_1-}, 0) \leq X_{T_1-}} \right] \\ &= \mathbb{E}_t \left[\int_t^{T_1-} e^{-r(s-t)} (Y_s - \eta_s X_s) ds \right] + \mathbb{E}_t \left[e^{-r(T_1-t)} P(Z_{T_1} Y_{T_1-}, 0) \mathbf{1}_{P(Z_{T_1} Y_{T_1-}, 0) > X_{T_1-}} \right] \\ &\quad + \mathbb{E}_t \left[e^{-r(T_1-t)} X_{T_1-} \mathbf{1}_{P(Z_{T_1} Y_{T_1-}, 0) \leq X_{T_1-}} \right] \\ &= \mathbb{E}_t \left[\int_t^{T_1-} e^{-r(s-t)} (Y_s - \eta_s X_s) ds \right] + \mathbb{E}_t \left[e^{-r(T_1-t)} P(Z_{T_1} Y_{T_1-}, 0) \right] \\ &\quad + \mathbb{E}_t \left[e^{-r(T_1-t)} (X_{T_1-} - P(Z_{T_1} Y_{T_1-}, 0)) \mathbf{1}_{P(Z_{T_1} Y_{T_1-}, 0) \leq X_{T_1-}} \right], \end{aligned} \quad (\text{IB.23})$$

where the second equality uses $P(Y, X) = P(Y, 0) - X$ for any $X < P(Y, 0)$ and $P(Y, X) = 0$ for any $X \geq P(Y, 0)$. Using (A.6), we have

$$\begin{aligned}
& \mathbb{E}_t \left[\int_t^{T_1-} e^{-r(s-t)} \eta_s X_s ds \right] = \mathbb{E}_t \left[\int_t^{T_1} e^{-r(s-t)} \eta_{s-} X_{s-} ds \right] \\
&= \mathbb{E}_t \left[\int_t^{T_1} e^{-r(s-t)} \lambda \mathbb{E}_{s-} \left[(X_{s-} - \ell Z_s Y_{s-}) \mathbf{1}_{\bar{x} Z_s Y_{s-} \leq X_{s-}} \right] ds \right] \\
&\geq \lambda \mathbb{E}_t \left[\int_t^{T_1} e^{-r(s-t)} (X_{s-} - \ell Z_s Y_{s-}) \mathbf{1}_{P(Z_s Y_{s-}, 0) \leq X_{s-}} ds \right] \\
&= \lambda \mathbb{E}_t \left[\int_t^\infty \int_t^T e^{-r(s-t)} \lambda e^{-\lambda(T-t)} (X_{s-} - \ell Z_s Y_{s-}) \mathbf{1}_{P(Z_s Y_{s-}, 0) \leq X_{s-}} ds dT \right] \\
&= \lambda \mathbb{E}_t \left[\int_t^\infty \int_s^\infty e^{-r(s-t)} \lambda e^{-\lambda(T-t)} dT (X_{s-} - \ell Z_s Y_{s-}) \mathbf{1}_{P(Z_s Y_{s-}, 0) \leq X_{s-}} ds \right] \\
&= \lambda \mathbb{E}_t \left[\int_t^\infty e^{-(r+\lambda)(s-t)} (X_{s-} - \ell Z_s Y_{s-}) \mathbf{1}_{P(Z_s Y_{s-}, 0) \leq X_{s-}} ds \right] \\
&= \mathbb{E}_t \left[e^{-r(T_1-t)} (X_{T_1-} - \ell Z_{T_1} Y_{T_1-}) \mathbf{1}_{P(Z_{T_1} Y_{T_1-}, 0) \leq X_{T_1-}} \right], \tag{IB.24}
\end{aligned}$$

where the inequality uses $\mathcal{A}(\bar{x}) \geq \bar{x} > \frac{1}{r-g} > \ell$ and $P(Z_s Y_{s-}, 0) = \mathcal{A}(\bar{x}) Z_s Y_{s-}$. Using (IB.24), we have

$$\begin{aligned}
& \mathbb{E}_t \left[\int_t^{T_1-} e^{-r(s-t)} (-\eta_s X_s) ds \right] + \mathbb{E}_t \left[e^{-r(T_1-t)} (X_{T_1-} - P(Z_{T_1} Y_{T_1-}, 0)) \mathbf{1}_{P(Z_{T_1} Y_{T_1-}, 0) \leq X_{T_1-}} \right] \\
&\leq \mathbb{E}_t \left[e^{-r(T_1-t)} (\ell Z_{T_1} Y_{T_1-} - P(Z_{T_1} Y_{T_1-}, 0)) \mathbf{1}_{P(Z_{T_1} Y_{T_1-}, 0) \leq X_{T_1-}} \right] \leq 0, \tag{IB.25}
\end{aligned}$$

where the first inequality uses (IB.24) and the second inequality uses (IB.21).

Substituting $P(Y, 0) = \mathcal{A}(\bar{x})Y$ and (IB.25) into (IB.23), we obtain

$$\mathcal{A}(\bar{x})Y_t \leq \mathbb{E}_t \left[\int_t^{T_1-} e^{-r(s-t)} Y_s ds \right] + \mathbb{E}_t \left[e^{-r(T_1-t)} \mathcal{A}(\bar{x}) Z_{T_1} Y_{T_1-} \right]. \tag{IB.26}$$

We have

$$\begin{aligned}
& \mathbb{E}_t \left[\int_t^{T_1-} e^{-r(s-t)} Y_s ds \right] \\
&= \int_t^\infty \int_t^T \lambda e^{-\lambda(T-t)} e^{-(r-\mu)(s-t)} Y_t ds dT \\
&= \int_t^\infty \int_s^\infty \lambda e^{-\lambda(T-t)} dT e^{-(r-\mu)(s-t)} Y_t ds \\
&= \int_t^\infty e^{-\lambda(s-t)} e^{-(r-\mu)(s-t)} Y_t ds = \frac{Y_t}{r + \lambda - \mu}, \tag{IB.27}
\end{aligned}$$

where $r + \lambda - \mu = r - g + \lambda \mathbb{E}[Z] > 0$. In addition, we have

$$\begin{aligned} & \mathbb{E}_t[e^{-r(T_1-t)} \mathcal{A}(\bar{x}) Z_{T_1} Y_{T_1-}] \\ &= \mathcal{A}(\bar{x}) \mathbb{E}[Z] Y_t \int_t^\infty \lambda e^{-\lambda(T-t)} e^{-(r-\mu)(T-t)} dT = \mathcal{A}(\bar{x}) \lambda \mathbb{E}[Z] \frac{Y_t}{r + \lambda - \mu}. \end{aligned} \quad (\text{IB.28})$$

Substituting (IB.27) and (IB.28) into (IB.26), we obtain

$$\mathcal{A}(\bar{x}) \leq \frac{1}{r + \lambda - \mu} + \mathcal{A}(\bar{x}) \lambda \mathbb{E}[Z] \frac{1}{r + \lambda - \mu}. \quad (\text{IB.29})$$

It follows that $\mathcal{A}(\bar{x}) \leq \frac{1}{r-g}$. Using $\mathcal{A}(\bar{x}) \geq \bar{x}$ when $\bar{x} > \frac{1}{r-g}$, we arrive at a contradiction. Thus, $\mathcal{A}(\bar{x}) < \bar{x}$ when $\bar{x} > \frac{1}{r-g}$. ■

Proof of Lemma 4. Let $\tilde{p}(x; \tilde{x}, \bar{x})$ denote the equity value under a target debt-EBIT ratio $\tilde{x}$ and a conjectured default boundary $\bar{x}$ by creditors, where $\bar{x} \geq \tilde{x} \geq 0$ and $x \geq 0$. Let $\tilde{v}(\tilde{x}, \bar{x}) = \inf\{x > 0 : \tilde{p}(x; \tilde{x}, \bar{x}) = 0\}$. Since $h_0 = h_1 = 0$, we have $\tilde{p}_x(x; \tilde{x}, \bar{x}) = -1$ for $x \leq \tilde{v}(\tilde{x}, \bar{x})$, and the firm value $\tilde{p}(x; \tilde{x}, \bar{x}) + x$ is independent of x . As a result, $\tilde{p}(x; \tilde{x}, \bar{x}) + x = [\tilde{p}(x; \tilde{x}, \bar{x}) + x]|_{x=\tilde{v}(\tilde{x}, \bar{x})} = \tilde{v}(\tilde{x}, \bar{x})$ for any $x \leq \tilde{v}(\tilde{x}, \bar{x})$.

In what follows, we simplify the notation by writing $\tilde{p}(x; \tilde{x}, \bar{x})$ as $\tilde{p}(x)$, and $\tilde{v}(\tilde{x}, \bar{x})$ as $\tilde{v}$. Substituting $\tilde{p}_x(x) = -1$ and $\tilde{p}_{xx}(x) = 0$ into the integro-differential equation (21) which holds at $x = \tilde{x}$ for $\tilde{p}(x)$, we obtain:

$$(r - \mu) \tilde{p}(\tilde{x}) = -[c(\tilde{x}; \bar{x}) + \theta(c(\tilde{x}; \bar{x})) - 1 - \mu \tilde{x}] + \lambda \mathbb{E}[Z \tilde{p}(\tilde{x}/Z) - \tilde{p}(\tilde{x})], \quad (\text{IB.30})$$

where $c(x; \bar{x}) = rx + \lambda \int_0^{x/\bar{x}} (x - \ell z) dF(z)$ represents the expected interest payment given the conjectured default boundary $\bar{x}$ of creditors. For any $x \leq \tilde{v}$,

$$\begin{aligned} & \mathbb{E}[Z \tilde{p}(x/Z) - \tilde{p}(x)] \\ &= \int_{x/\tilde{v}}^1 [z \tilde{p}(x/z) - \tilde{p}(x)] dF(z) + \int_0^{x/\tilde{v}} (-\tilde{p}(x)) dF(z) \\ &= \int_{x/\tilde{v}}^1 [z \tilde{v} - \tilde{v}] dF(z) + \int_0^{x/\tilde{v}} (x - \tilde{v}) dF(z) \\ &= \int_0^1 [z - 1] \tilde{v} dF(z) - \int_0^{x/\tilde{v}} [z - 1] \tilde{v} dF(z) + \int_0^{x/\tilde{v}} (x - \tilde{v}) dF(z) \\ &= \mathbb{E}[z - 1] \tilde{v} + \int_0^{x/\tilde{v}} (x - z \tilde{v}) dF(z), \end{aligned} \quad (\text{IB.31})$$

where the first and second equalities use $\tilde{p}(x) + x = \tilde{v}$ for any $x \leq \tilde{v}$ and $\tilde{p}(x) = 0$ for $x > \tilde{v}$. Substituting (IB.31) into (IB.30), and using $\tilde{p}(\tilde{x}) = \tilde{v} - \tilde{x}$, we have

$$(r - g) \tilde{v} = r \tilde{x} + 1 - \theta(c(\tilde{x}; \bar{x})) - c(\tilde{x}; \bar{x}) + \lambda \int_0^{\tilde{x}/\tilde{v}} (\tilde{x} - z \tilde{v}) dF(z). \quad (\text{IB.32})$$

Then we obtain equation (IB.8) about $\tilde{v} \geq \tilde{x}$.

Denote the left side of (IB.8) as $\mathcal{R}(\tilde{x}, \tilde{v})$, and the right side of (IB.8) as $\mathcal{H}(\tilde{x}, \bar{x})$. For any $\tilde{v} \geq \tilde{x} \geq 0$, we have the following partial derivatives:⁵¹

$$\mathcal{R}_{\tilde{x}}(\tilde{x}, \tilde{v}) = -\lambda F\left(\frac{\tilde{x}}{\tilde{v}}\right), \quad (\text{IB.33})$$

$$\mathcal{R}_{\tilde{v}}(\tilde{x}, \tilde{v}) = r - g + \lambda \int_0^{\tilde{x}/\tilde{v}} z dF(z), \quad (\text{IB.34})$$

$$\mathcal{H}_{\tilde{x}}(\tilde{x}, \bar{x}) = r - [\mathbf{1}_{c(\tilde{x}; \bar{x}) \leq 1}(1 - \tau) + \mathbf{1}_{c(\tilde{x}; \bar{x}) > 1}] \left[r + \lambda \frac{\tilde{x}}{\bar{x}} \left(1 - \frac{\ell}{\bar{x}}\right) F'\left(\frac{\tilde{x}}{\bar{x}}\right) + \lambda F\left(\frac{\tilde{x}}{\bar{x}}\right) \right], \quad (\text{IB.35})$$

$$\mathcal{H}_{\bar{x}}(\tilde{x}, \bar{x}) = [\mathbf{1}_{c(\tilde{x}; \bar{x}) \leq 1}(1 - \tau) + \mathbf{1}_{c(\tilde{x}; \bar{x}) > 1}] \lambda \frac{\tilde{x}^2}{\bar{x}^2} \left(1 - \frac{\ell}{\bar{x}}\right) F'\left(\frac{\tilde{x}}{\bar{x}}\right). \quad (\text{IB.36})$$

Since $\lim_{\tilde{v} \rightarrow \infty} \mathcal{R}(\tilde{x}, \tilde{v}) = \infty$ and $\mathcal{R}_{\tilde{v}}(\tilde{x}, \tilde{v}) \geq r - g > 0$, equation (IB.8) admits a unique solution $\tilde{v} \geq \tilde{x}$ if and only if $\mathcal{R}(\tilde{x}, \tilde{x}) \leq \mathcal{H}(\tilde{x}, \bar{x})$. We define the feasible set of target debt-EBIT ratios $\tilde{x}$ as follows:

$$\mathbb{U}(\bar{x}) = \{\tilde{x} \in [0, \bar{x}] : \mathcal{R}(\tilde{x}, \tilde{x}) \leq \mathcal{H}(\tilde{x}, \bar{x})\}. \quad (\text{IB.37})$$

Since $\mathcal{R}(\tilde{x}, \tilde{x}) \leq \mathcal{H}(\tilde{x}, \bar{x})$ is equivalent to $1 + \mu\tilde{x} - \theta(c(\tilde{x}; \bar{x})) - c(\tilde{x}; \bar{x}) \geq 0$, we obtain (IB.7).

The set $\mathbb{U}(\bar{x})$ is nonempty because $[0, \tilde{x}_1] \subset \mathbb{U}(\bar{x})$, where $\tilde{x}_1 = \inf\{\tilde{x} \in [0, \bar{x}] : 1 + \mu\tilde{x} - \theta(c(\tilde{x}; \bar{x})) - c(\tilde{x}; \bar{x}) \leq 0\} > 0$ since $c(0; \bar{x}) = 0$, and $\theta(0) = \tau < 1$. Moreover, since $c(\tilde{x}; \bar{x})$ is decreasing in $\bar{x}$, $\theta(c(\tilde{x}; \bar{x})) + c(\tilde{x}; \bar{x}) = \mathbf{1}_{c(\tilde{x}; \bar{x}) \leq 1}[\tau + c(\tilde{x}; \bar{x})(1 - \tau)] + c(\tilde{x}; \bar{x})\mathbf{1}_{c(\tilde{x}; \bar{x}) > 1}$ is also decreasing in $\bar{x}$. It follows that $\mathbb{U}(\bar{x})$ is larger for larger $\bar{x}$: $\mathbb{U}(\bar{x}) \subseteq \mathbb{U}(\bar{x} + \Delta x)$ for $\Delta x > 0$.

In sum, for any $\bar{x} \geq \ell$ and $\tilde{x} \in \mathbb{U}(\bar{x})$, firm value $\tilde{v}(\tilde{x}, \bar{x})$ is uniquely determined by

$$\mathcal{R}(\tilde{x}, \tilde{v}(\tilde{x}, \bar{x})) = \mathcal{H}(\tilde{x}, \bar{x}). \quad (\text{IB.38})$$

For any $\tilde{x} \in [0, \bar{x}] \setminus \mathbb{U}(\bar{x})$ and $\tilde{v} \geq \tilde{x}$, $\mathcal{R}(\tilde{x}, \tilde{v}) \geq \mathcal{R}(\tilde{x}, \tilde{x}) > \mathcal{H}(\tilde{x}, \bar{x})$, and thus (IB.8) has no solution $\tilde{v} \geq \tilde{x}$. It follows that the strategy S which maintains $X_s/Y_s = \tilde{x}$ is not feasible given creditors' belief $\tilde{S}$ with a default region $\{(Y, X) : X/Y > \bar{x}\}$.

Using (IB.34) and (IB.36), we have $\mathcal{R}_{\tilde{v}}(\tilde{x}, \tilde{v}) > 0$ and $\mathcal{H}_{\bar{x}}(\tilde{x}, \bar{x}) > 0$. Then it follows from (IB.38) that $\tilde{v}(\tilde{x}, \bar{x})$ is increasing in $\bar{x} \geq \ell$. For any $\Delta x > 0$, and $\tilde{x} \in \mathbb{U}(\bar{x}) \subseteq \mathbb{U}(\bar{x} + \Delta x)$, $\mathcal{A}(\bar{x} + \Delta x) \geq \tilde{v}(\tilde{x}, \bar{x} + \Delta x) > \tilde{v}(\tilde{x}, \bar{x})$, which implies that $\mathcal{A}(\bar{x} + \Delta x) > \max_{\tilde{x} \in \mathbb{U}(\bar{x})} \tilde{v}(\tilde{x}, \bar{x}) = \mathcal{A}(\bar{x})$. That is, $\mathcal{A}(\bar{x})$ is increasing in $\bar{x} \geq \ell$.

According to Lemma 3, the firm's optimal strategy is to maintain X_t/Y_t at a constant level. Since $\tilde{v}(\tilde{x}, \bar{x})$ denotes the firm value under a target debt-EBIT ratio $\tilde{x}$ and the conjectured default boundary $\bar{x}$ by creditors, we then obtain (IB.9).

⁵¹ For $\tilde{x}$ satisfying $c(\tilde{x}; \bar{x}) = 1$, we have $c(x; \bar{x}) < 1$ for $x < \tilde{x}$, and $\mathcal{H}_{\bar{x}}(\tilde{x}, \bar{x})$ refers to the left partial derivative $\mathcal{H}_{\bar{x}}(\tilde{x} -, \bar{x})$. Similarly, we have $c(\tilde{x}; x) < 1$ for $x > \bar{x}$, and $\mathcal{H}_{\bar{x}}(\tilde{x}, \bar{x})$ refers to the right partial derivative $\mathcal{H}_{\bar{x}}(\tilde{x}, \bar{x} +)$.

For each $\bar{x} \geq \tilde{x} \geq 0$, since $\tilde{v} = \tilde{v}(\tilde{x}, \bar{x})$ satisfies equation (IB.8), when $\tilde{v} = \tilde{v}(\tilde{x}, \bar{x}) \geq \bar{x} \geq \ell$, we have

$$\begin{aligned}
(r-g)\tilde{v} &= \lambda \int_0^{\tilde{x}/\tilde{v}} (\tilde{x} - z\tilde{v}) dF(z) + \mathcal{H}(\tilde{x}, \bar{x}) \\
&\leq \lambda \int_0^{\tilde{x}/\tilde{v}} (\tilde{x} - z\tilde{v}) dF(z) + \mathcal{H}(\tilde{x}, \tilde{v}) \\
&= \lambda \int_0^{\tilde{x}/\tilde{v}} (\tilde{x} - z\tilde{v}) dF(z) + 1 - \theta(c(\tilde{x}; \tilde{v})) - \lambda \int_0^{\tilde{x}/\tilde{v}} (\tilde{x} - \ell z) dF(z) \\
&= 1 - \theta(c(\tilde{x}; \tilde{v})) + \lambda \int_0^{\tilde{x}/\tilde{v}} (\ell z - z\tilde{v}) dF(z) \leq 1,
\end{aligned} \tag{IB.39}$$

where the first inequality uses $\mathcal{H}_{\bar{x}}(\tilde{x}, \bar{x}) \geq 0$ for $\bar{x} \geq \ell$, the second inequality uses $\tilde{v} \geq \ell$.

Thus, we derive from (IB.39) that

$$\tilde{v}(\tilde{x}, \bar{x}) \leq \frac{1}{r-g} \quad \text{if } \tilde{v}(\tilde{x}, \bar{x}) \geq \bar{x}. \tag{IB.40}$$

Using (IB.40), we can see that $\tilde{v}(\tilde{x}, \bar{x}) < \bar{x}$ for any $\bar{x} > \frac{1}{r-g}$, $\tilde{x} \in \mathbb{U}(\bar{x})$. ■

Proof of Theorem 5.

Step 1: We prove that there exists $\bar{x} \in (\bar{x}_0, \bar{x}_1)$, such that $\mathcal{A}(\bar{x}) = \bar{x}$ if $\mathcal{A}(\bar{x}_0) > \bar{x}_0$ and $\mathcal{A}(\bar{x}_1) < \bar{x}_1$.

Let $\mathbb{A} = \{x \in [\bar{x}_0, \bar{x}_1] : \mathcal{A}(x) \geq x\}$. Since $\mathcal{A}(\bar{x}_0) > \bar{x}_0$, $\mathbb{A}$ is nonempty.

Let $\bar{x} = \sup \mathbb{A}$. We have $\bar{x}_0 \leq \bar{x} \leq \bar{x}_1$. If $\bar{x} \notin \mathbb{A}$, then $\mathcal{A}(\bar{x}) < \bar{x}$. By the definition of $\bar{x} = \sup \mathbb{A}$ and $\bar{x} \notin \mathbb{A}$, there exists an increasing sequence $x_n \in \mathbb{A}$ such that $\lim_{n \rightarrow \infty} x_n = \bar{x}$. Thus $\mathcal{A}(x_n) \geq x_n$. Recall that we have shown that $\mathcal{A}(x)$ is increasing in x in the proof of Lemma 4. Then we have

$$\mathcal{A}(\bar{x}) \geq \mathcal{A}(x_n) \geq x_n. \tag{IB.41}$$

Sending n to infinity in the above inequality, we obtain $\mathcal{A}(\bar{x}) \geq \lim_{n \rightarrow \infty} x_n = \bar{x}$. Then we arrive at a contradiction.

Thus $\bar{x} \in \mathbb{A}$, and then $\mathcal{A}(\bar{x}) \geq \bar{x}$. Since $\mathcal{A}(\bar{x}_1) < \bar{x}_1$, we have $\bar{x} < \bar{x}_1$. By the definition of $\bar{x} = \sup \mathbb{A}$, for any small $\varepsilon \in (0, \bar{x}_1 - \bar{x})$, we have $\bar{x} + \varepsilon \notin \mathbb{A}$ and then $\mathcal{A}(\bar{x} + \varepsilon) < \bar{x} + \varepsilon$. Since $\mathcal{A}(x)$ is increasing in x , we have

$$\bar{x} \leq \mathcal{A}(\bar{x}) \leq \mathcal{A}(\bar{x} + \varepsilon) < \bar{x} + \varepsilon. \tag{IB.42}$$

Sending ε to 0 in the above inequality, we obtain $\mathcal{A}(\bar{x}) = \bar{x}$.

Step 2: We show that there exist $\bar{x}^* > \ell$ and $\tilde{x}^* \in \mathbb{U}(\bar{x}^*)$, such that $\tilde{v}(\tilde{x}^*, \bar{x}^*) = \bar{x}^* \geq \tilde{v}(\tilde{x}, \bar{x}^*)$ for any $\tilde{x} \in \mathbb{U}(\bar{x}^*)$.

Using Lemma 3, we have $\mathcal{A}(\bar{x}) < \bar{x}$ for any $\bar{x} > \frac{1}{r-g}$. In addition, since $\mathcal{R}(0, \tilde{v}) = (r-g)\tilde{v}$ and $\mathcal{H}(0, \bar{x}) = 1 - \tau$, we derive from (IB.38) that $\tilde{v}(0, \bar{x}) = \frac{1-\tau}{r-g} > \ell$ under Assumption 1. It follows

that $\mathcal{A}(\bar{x}) \geq \tilde{v}(0, \bar{x}) > \ell$ for any $\bar{x} \geq \ell$. Using $\mathcal{A}(\ell) - \ell > 0$, $\mathcal{A}(\bar{x}) - \bar{x} < 0$ for any $\bar{x} > \frac{1}{r-g}$, and the result proved in step 1, we conclude that there exists $\bar{x}^* \in (\ell, \frac{1}{r-g}]$, such that $\mathcal{A}(\bar{x}^*) = \bar{x}^*$. Let $\tilde{x}^* = \arg \max_{\tilde{x} \in \mathbb{U}(\bar{x}^*)} \tilde{v}(\tilde{x}, \bar{x}^*)$. Then we have $\tilde{v}(\tilde{x}^*, \bar{x}^*) = \mathcal{A}(\bar{x}^*) = \bar{x}^* \geq \tilde{v}(\tilde{x}, \bar{x}^*)$ for any $\tilde{x} \in \mathbb{U}(\bar{x}^*)$.

Step 3: We prove that $\mathcal{L}^{\bar{x}^} \tilde{p}(x) \leq 0$ for any $x \in [0, \bar{x}^*]$, where $\tilde{p}(x) = \tilde{p}(x; \tilde{x}^*, \bar{x}^*)$ is the equity value under a target debt-EBIT ratio $\tilde{x}^*$ and a conjectured default boundary $\bar{x}^*$ by creditors, and $\mathcal{L}^{\bar{x}}$ is defined in (B.7).*

Since $\tilde{p}(x) + x = \tilde{v}(\tilde{x}^*, \bar{x}^*) = \bar{x}^*$ for any $x \in [0, \bar{x}^*]$, we have

$$\begin{aligned} & \mathcal{L}^{\bar{x}^*} \tilde{p}(x) \\ &= (r - \mu)x - (r - \mu)\bar{x}^* - [c(x; \bar{x}^*) + \theta(c(x; \bar{x}^*)) - 1 - \mu x] + \lambda \mathbb{E}[Z\tilde{p}(x/Z) - \tilde{p}(x)] \\ &= rx + 1 - c(x; \bar{x}^*) - \theta(c(x; \bar{x}^*)) - (r - \mu)\bar{x}^* + \lambda \left(\mathbb{E}[Z - 1]\bar{x}^* + \int_0^{x/\bar{x}^*} (x - z\bar{x}^*) dF(z) \right) \\ &= \mathcal{H}(x, \bar{x}^*) - \mathcal{R}(x, \bar{x}^*), \end{aligned} \tag{IB.43}$$

where the second equality uses (IB.31) and $\tilde{v}(\tilde{x}^*, \bar{x}^*) = \bar{x}^*$, and the third equality uses $g = \mu - \lambda(1 - \mathbb{E}(Z))$. Using (IB.37), we conclude that for any $x \in [0, \bar{x}^*] \setminus \mathbb{U}(\bar{x}^*)$ and $\tilde{v} \geq x$, $\mathcal{R}(x, \tilde{v}) \geq \mathcal{R}(x, x) > \mathcal{H}(x, \bar{x}^*)$. In particular, $\mathcal{R}(x, \bar{x}^*) > \mathcal{H}(x, \bar{x}^*)$ for any $x \in [0, \bar{x}^*] \setminus \mathbb{U}(\bar{x}^*)$.

Using the result proved in step 2, for any $x \in \mathbb{U}(\bar{x}^*)$, we have $\tilde{v}(x, \bar{x}^*) \leq \mathcal{A}(\bar{x}^*) = \bar{x}^*$. Since $\mathcal{R}_v(x, v) > 0$, we have $\mathcal{R}(x, \bar{x}^*) \geq \mathcal{R}(x, \tilde{v}(x, \bar{x}^*)) = \mathcal{H}(x, \bar{x}^*)$ for any $x \in \mathbb{U}(\bar{x}^*)$. Then it follows from (IB.43) that $\mathcal{L}^{\bar{x}^*} \tilde{p}(x) = \mathcal{H}(x, \bar{x}^*) - \mathcal{R}(x, \bar{x}^*) \leq 0$ for any $x \in [0, \bar{x}^*]$.

Since $\tilde{p}(x) = \bar{x}^* - x$, for $x \in [0, \bar{x}^*]$, and $\bar{x}^* > \ell$, we have $\tilde{p}(x) \geq \max\{\ell - x, 0\}$, $\tilde{p}_x(x) = -1$ for $x \in [0, \bar{x}^*]$. Combining these with $\mathcal{L}^{\bar{x}^*} \tilde{p}(x) \leq 0$ for any $x \in [0, \bar{x}^*]$, we can see that $\tilde{p}(x)$ satisfies (IA.32) for $x \in [0, \bar{x}^*]$. In addition, note that $\tilde{p}(x) = 0$ for $x \geq \bar{x}^*$ and $\tilde{p}(x) > 0$ for $x \in [0, \bar{x}^*)$.

Let $\tilde{P}(Y, X) = Y\tilde{p}(X/Y)$, and let S denote the strategy with target debt-EBIT ratio $\tilde{x}^*$, and default boundary $\bar{x}^*$. We can see that $\tilde{P}(Y, X) = \mathcal{P}(Y, X; S, S)$ and $\tilde{P}(Y, X)$ satisfies all conditions in Theorem 1. Hence, S is an MPE.

Step 4: We prove the existence of the best equilibrium.

For any MPE $S \in \mathcal{S}_x$, its default region is given by $\mathcal{D} = \{(Y, X) : X/Y > \bar{x}\}$ for some $\bar{x} \geq \ell$. Combining with Lemma 3, we can see any MPE $S \in \mathcal{S}_x$ is characterized by a default boundary $\bar{x} > \ell$ and a recapitalization target $\tilde{x}$. As a result, the set of default boundaries in all equilibria $\in \mathcal{S}_x$ can be represented by

$$\mathbb{S}^* = \{\bar{x} > \ell : \mathcal{A}(\bar{x}) = \bar{x}\}. \tag{IB.44}$$

By applying Lemma 3, we obtain $\mathcal{A}(\bar{x}) < \bar{x}$ for any $\bar{x} > \frac{1}{r-g}$. It follows that $\bar{x} \leq \frac{1}{r-g}$ for any $\bar{x} \in \mathbb{S}^*$, and then $\bar{x}^* = \sup \mathbb{S}^* \leq \frac{1}{r-g}$.

Next, we prove that $\bar{x}^* \in \mathbb{S}^*$. For the sake of contradiction, assume that $\bar{x}^* \notin \mathbb{S}^*$ and then $\mathcal{A}(\bar{x}^*) \neq \bar{x}^*$. Then there exists a sequence $\{\bar{x}_n\}_{n \geq 1}$ in the set $\mathbb{S}^*$, such that $\bar{x}_n < \bar{x}^*$ and $\lim_{n \rightarrow \infty} \bar{x}_n = \bar{x}^*$. Because $\mathcal{A}(\cdot)$ is a nondecreasing function, we have $\mathcal{A}(\bar{x}^*) \geq \mathcal{A}(\bar{x}_n) = \bar{x}_n$. Sending n to infinity, we have $\mathcal{A}(\bar{x}^*) \geq \bar{x}^*$. Using $\mathcal{A}(\bar{x}^*) \neq \bar{x}^*$, we have $\mathcal{A}(\bar{x}^*) > \bar{x}^*$. Recalling Lemma 3

implies that $\mathcal{A}(\bar{x}) < \bar{x}$ for any $\bar{x} > \frac{1}{r-g}$, and using the result proved in step 1, we conclude that for $\bar{x}_2 > \frac{1}{r-g}$, there exists $\bar{x}_1 \in (\bar{x}^*, \bar{x}_2)$, such that $\mathcal{A}(\bar{x}_1) = \bar{x}_1$ so that $\bar{x}_1 \in \mathbb{S}^*$. Then we arrive at a contradiction as $\bar{x}^* = \sup \mathbb{S}^*$ and $\bar{x}_1 > \bar{x}^*$.

Therefore, we have $\bar{x}^* \in \mathbb{S}^*$. Let $\tilde{x}^* = \arg \max_{\tilde{x} \in \mathbb{U}(\bar{x}^*)} \tilde{v}(\tilde{x}, \bar{x}^*)$. Then the strategy S^* with target debt-EBIT ratio $\tilde{x}^*$, and default boundary $\bar{x}^*$ is the MPE with the highest firm value $\bar{x}^*$. ■

Proof of Proposition 6. Write

$$a = r - g > 0, \quad I(z) = \int_0^z u dF(u), \quad B(z) = 1 + \lambda I(z),$$

and define

$$H(z) = z[r + \lambda F(z)] - \lambda I(z) = rz + \lambda \int_0^z (z - u) dF(u).$$

Since $H'(z) = r + \lambda F(z) > 0$, H is strictly increasing, with $H(0) = 0$ and $H(1) = a + \mu$.

Step 1: Characterization of any feasible pair.

Consider any pair $(\tilde{x}, \bar{x})$ satisfying $\bar{x} \geq \tilde{x} \geq 0$ and (IB.3), and let $z = \tilde{x}/\bar{x} \in [0, 1]$. Debt pricing implies $c(\tilde{x}; \bar{x}) \geq 0$, so $0 \leq \theta(c(\tilde{x}; \bar{x})) \leq \tau$. Substituting $\theta(c(\tilde{x}; \bar{x})) \leq \tau$ into (IB.3), we have

$$\begin{aligned} \bar{x} &\geq \frac{1}{r-g} \left[1 - \tau - \lambda(\bar{x} - \ell) \left(\int_0^{\tilde{x}/\bar{x}} z dF(z) \right) \right] \\ &> \ell + \frac{1}{r-g} \left[-\lambda(\bar{x} - \ell) \left(\int_0^{\tilde{x}/\bar{x}} z dF(z) \right) \right], \end{aligned} \quad (\text{IB.45})$$

where the second inequality uses Assumption 1. It follows that $\bar{x} > \ell$. Since $\bar{x} > \ell$, it follows from (IB.3) that $\bar{x} \leq \frac{1}{r-g}$. Thus, for any $\tilde{x} \in \tilde{\mathbb{U}}$ and $\bar{x} \geq \tilde{x}$ satisfying (IB.3), we have

$$\ell < \bar{x} \leq \frac{1}{r-g}. \quad (\text{IB.46})$$

Because $\bar{x} > \ell$, the interest payment is

$$c := c(\tilde{x}; \bar{x}) = \bar{x}z[r + \lambda F(z)] - \lambda I(z).$$

Combining this expression with (IB.3) yields

$$c - 1 = \bar{x}[H(z) - a] - \theta(c).$$

Using $\theta(c) = \tau(1 - c)^+$, we obtain

$$\bar{x}[H(z) - a] = \begin{cases} (1 - \tau)(c - 1), & c \leq 1, \\ c - 1, & c \geq 1. \end{cases}$$

Hence $c - 1$ and $H(z) - a$ have the same sign.

If $\mu \leq 0$, then $H(z) \leq a$ throughout $[0, 1]$, so $c \leq 1$. If $\mu > 0$, there is a unique $z_0 \in (0, 1)$ satisfying $H(z_0) = a$; in this case, $c \leq 1$ if and only if $z \leq z_0$.

Substituting the corresponding tax branch into (IB.3) gives

$$\bar{x} = \begin{cases} f_1(z) = \frac{(1-\tau)B(z)}{a - \tau z[r + \lambda F(z)] + \lambda I(z)}, & H(z) \leq a, \\ f_2(z) = \frac{B(z)}{a + \lambda I(z)}, & H(z) \geq a. \end{cases}$$

Therefore every feasible pair satisfies $\bar{x} = f_0(z)$ and $\tilde{x} = z f_0(z)$, with f_0 defined in (IB.13).

Step 2: Every point on the parameterized curve is feasible.

Let

$$D_1(z) = a - \tau z[r + \lambda F(z)] + \lambda I(z), \quad D_2(z) = a + \lambda I(z) > 0.$$

On the relevant domain $H(z) \leq a$ of f_1 ,

$$D_1(z) = a - \tau H(z) + (1 - \tau)\lambda I(z) \geq (1 - \tau)[a + \lambda I(z)] > 0.$$

Thus both branches are well defined on their respective domains. Moreover,

$$f_1(z) - \ell = \frac{1 - \tau - a\ell + \tau\ell H(z)}{D_1(z)} > 0, \quad f_2(z) - \ell = \frac{1 - a\ell}{D_2(z)} > 0,$$

using $H(z) \geq 0$ and $a\ell < 1 - \tau$.

For each $z \in [0, 1]$, set $\bar{x} = f_0(z)$ and $\tilde{x} = z f_0(z)$. Then $\bar{x} > \ell$ and $0 \leq \tilde{x} \leq \bar{x}$. Direct calculation gives, on the relevant domain of each branch,

$$c(z f_i(z); f_i(z)) - 1 = \frac{B(z)}{D_i(z)}[H(z) - a], \quad i = 1, 2.$$

Consequently, each candidate satisfies the tax restriction used to derive its branch. Substitution into (IB.3) therefore verifies that the candidate pair is feasible. Together with Step 1, this establishes

$$\tilde{\mathcal{U}} = \{z f_0(z) : z \in [0, 1]\}.$$

Step 3: The feasible set is a closed interval.

Each branch is continuous on its relevant domain. When $\mu > 0$, $H(z_0) = a$ implies

$$D_1(z_0) = (1 - \tau)D_2(z_0),$$

and hence $f_1(z_0) = f_2(z_0)$. Thus f_0 is continuous and strictly positive on $[0, 1]$.

The function $z \mapsto z f_0(z)$ is therefore continuous, nonnegative, and equal to zero at $z = 0$. Its image is consequently

$$\tilde{\mathcal{U}} = \left[0, \max_{z \in [0, 1]} z f_0(z)\right] = [0, \tilde{x}_{\max}],$$

where the maximum is attained and is strictly positive. This proves (IB.11) and (IB.12). $\blacksquare$

Proof of Theorem 6. Since $f_0(z)$, as given in Proposition 6, is continuous in $z \in [0, 1]$, there exists $z_* \in [0, 1]$, such that (IB.14) holds with $\bar{x}_* = f_0(z_*)$.

Using (IB.46), we have $\ell < \bar{x}_* \leq \frac{1}{r-g}$. Recall that in step 4 of the proof of Theorem 5, we have shown that the strategy with target debt-EBIT ratio $\tilde{x}^*$, and default boundary $\bar{x}^*$ is the MPE with highest firm value $\bar{x}^*$. Moreover, we have shown that $\bar{x}^* \in \mathbb{S}^*$ given in (IB.44). Thus, $\bar{x}^* = \mathcal{A}(\bar{x}^*) = \tilde{v}(\tilde{x}^*, \bar{x}^*)$. Then $\tilde{x}^* \in \tilde{\mathbb{U}}$ and thus

$$\bar{x}^* = f_0(\tilde{x}^*/\bar{x}^*) \leq \bar{x}_*, \quad (\text{IB.47})$$

where the equality uses Proposition 6, and the inequality uses $\bar{x}_* = \max_{z \in [0, 1]} f_0(z)$.

By the definition of $\bar{x}_*$ and Proposition 6, we have $\tilde{v}(\tilde{x}_*, \bar{x}_*) = \bar{x}_*$. For the sake of contradiction, assume there exists $\tilde{x}_1 \in \mathbb{U}(\bar{x}_*)$ such that $\tilde{v}(\tilde{x}_1, \bar{x}_*) > \bar{x}_*$. Using (IB.40), we can see that $\tilde{v}(\tilde{x}_1, \bar{x}) < \bar{x}$ for any $\bar{x} > \frac{1}{r-g}$. Thus, there exists $\bar{x}_1 > \bar{x}_*$, such that $\tilde{v}(\tilde{x}_1, \bar{x}_1) = \bar{x}_1$. Consequently,

$$\bar{x}_1 = f_0(\tilde{x}_1/\bar{x}_1) \leq \bar{x}_*,$$

where the equality uses Proposition 6, and the inequality uses $\bar{x}_* = \max_{z \in [0, 1]} f_0(z)$. Then we arrive at a contradiction as $\bar{x}_1 > \bar{x}_*$. Therefore,

$$\tilde{v}(\tilde{x}, \bar{x}_*) \leq \bar{x}_* = \tilde{v}(\tilde{x}_*, \bar{x}_*) \quad (\text{IB.48})$$

for any $\tilde{x} \in \mathbb{U}(\bar{x}_*)$. Then using (IB.9) and (IB.48), we conclude that $\bar{x}_* \in \mathbb{S}^*$ given in (IB.44). Since we have shown that $\bar{x}^* = \sup\{\bar{x} : \bar{x} \in \mathbb{S}^*\}$ in the proof of Theorem 5, we have $\bar{x}_* \leq \bar{x}^*$. Combining with (IB.47), we have $\bar{x}_* = \bar{x}^*$.

Therefore, $\bar{x}_* = \bar{x}^*$ is the default boundary in the MPE with the highest firm value. Moreover, by (IB.48) the target $\tilde{x}_* = \bar{x}_* z_*$ attains $\mathcal{A}(\bar{x}_*) = \bar{x}_*$ for any maximum point z_* of f_0 , so the proof of Theorem 5 shows that the strategy with target $\tilde{x}_*$ and default boundary $\bar{x}_*$ is this best MPE. $\blacksquare$

Proof of Proposition 7-(i). For the sake of contradiction, assume there exist $1 \geq z_2 > z_1 \geq 0$, such that $\bar{x}_* = f_0(z_1) = f_0(z_2)$. Denote $\tilde{x}_1 = z_1 \bar{x}_*$, and $\tilde{x}_2 = z_2 \bar{x}_*$. Then by applying (IB.38) and Proposition 6, we obtain

$$\mathcal{R}(\tilde{x}_i, \bar{x}_*) = \mathcal{H}(\tilde{x}_i, \bar{x}_*), \quad (\text{IB.49})$$

where $i = 1, 2$, $\mathcal{R}(\tilde{x}, \bar{x})$ denotes the left side of (IB.8), and $\mathcal{H}(\tilde{x}, \bar{x})$ denotes the right side of (IB.8). We prove that

$$\mathcal{R}(\tilde{x}_0, \bar{x}_*) < \mathcal{H}(\tilde{x}_0, \bar{x}_*) \quad (\text{IB.50})$$

for some $\tilde{x}_0 \in (\tilde{x}_1, \tilde{x}_2)$.

Using (IB.33), (IB.35), we have

$$\begin{aligned} & \mathcal{H}_{\tilde{x}}(\tilde{x}, \bar{x}) - \mathcal{R}_{\tilde{x}}(\tilde{x}, \bar{x}) \\ &= r + \lambda F\left(\frac{\tilde{x}}{\bar{x}}\right) - [\mathbf{1}_{c(\tilde{x}; \bar{x}) \leq 1}(1 - \tau) + \mathbf{1}_{c(\tilde{x}; \bar{x}) > 1}] \left[r + \lambda \frac{\tilde{x}}{\bar{x}} \left(1 - \frac{\ell}{\bar{x}}\right) F'\left(\frac{\tilde{x}}{\bar{x}}\right) + \lambda F\left(\frac{\tilde{x}}{\bar{x}}\right) \right]. \end{aligned} \quad (\text{IB.51})$$

If $c(\tilde{x}_2; \bar{x}_*) > 1$, then there exists a sufficiently small $\varepsilon > 0$, such that $\mathcal{H}_{\tilde{x}}(\tilde{x}, \bar{x}_*) - \mathcal{R}_{\tilde{x}}(\tilde{x}, \bar{x}_*) = -\lambda \frac{\tilde{x}}{\bar{x}_*} \left(1 - \frac{\ell}{\bar{x}_*}\right) F'\left(\frac{\tilde{x}}{\bar{x}_*}\right) < 0$ for $\tilde{x} \in (\tilde{x}_2 - \varepsilon, \tilde{x}_2)$. It follows that $\mathcal{H}(\tilde{x}_0, \bar{x}_*) - \mathcal{R}(\tilde{x}_0, \bar{x}_*) > \mathcal{H}(\tilde{x}_2, \bar{x}_*) - \mathcal{R}(\tilde{x}_2, \bar{x}_*) = 0$ for $\tilde{x}_0 \in (\tilde{x}_2 - \varepsilon, \tilde{x}_2)$.

Thus, we only need to consider the case $c(\tilde{x}_2; \bar{x}_*) \leq 1$. Then we have $c(\tilde{x}; \bar{x}_*) < 1$ for any $\tilde{x} \in [\tilde{x}_1, \tilde{x}_2]$, and then

$$\mathcal{H}_{\tilde{x}}(\tilde{x}, \bar{x}_*) - \mathcal{R}_{\tilde{x}}(\tilde{x}, \bar{x}_*) = \tau r + \lambda \left[\tau F\left(\frac{\tilde{x}}{\bar{x}_*}\right) - (1 - \tau) \frac{\tilde{x}}{\bar{x}_*} \left(1 - \frac{\ell}{\bar{x}_*}\right) F'\left(\frac{\tilde{x}}{\bar{x}_*}\right) \right]. \quad (\text{IB.52})$$

If $\mathcal{H}_{\tilde{x}}(\tilde{x}_1, \bar{x}_*) - \mathcal{R}_{\tilde{x}}(\tilde{x}_1, \bar{x}_*) > 0$ or $\mathcal{H}_{\tilde{x}}(\tilde{x}_2, \bar{x}_*) - \mathcal{R}_{\tilde{x}}(\tilde{x}_2, \bar{x}_*) < 0$, then using (IB.49), we conclude that there exists $\tilde{x}_0 \in (\tilde{x}_1, \tilde{x}_2)$, such that $\mathcal{R}(\tilde{x}_0, \bar{x}_*) < \mathcal{H}(\tilde{x}_0, \bar{x}_*)$. Thus, we only need to consider the case $\mathcal{H}_{\tilde{x}}(\tilde{x}_1, \bar{x}_*) - \mathcal{R}_{\tilde{x}}(\tilde{x}_1, \bar{x}_*) \leq 0$ and $\mathcal{H}_{\tilde{x}}(\tilde{x}_2, \bar{x}_*) - \mathcal{R}_{\tilde{x}}(\tilde{x}_2, \bar{x}_*) \geq 0$. It follows that

$$\tau r + \lambda \left[\tau F\left(\frac{\tilde{x}_1}{\bar{x}_*}\right) - (1 - \tau) \frac{\tilde{x}_1}{\bar{x}_*} \left(1 - \frac{\ell}{\bar{x}_*}\right) F'\left(\frac{\tilde{x}_1}{\bar{x}_*}\right) \right] \leq 0, \quad (\text{IB.53})$$

$$\tau r + \lambda \left[\tau F\left(\frac{\tilde{x}_2}{\bar{x}_*}\right) - (1 - \tau) \frac{\tilde{x}_2}{\bar{x}_*} \left(1 - \frac{\ell}{\bar{x}_*}\right) F'\left(\frac{\tilde{x}_2}{\bar{x}_*}\right) \right] \geq 0. \quad (\text{IB.54})$$

Using (IB.53), we have

$$\tau F\left(\frac{\tilde{x}_1}{\bar{x}_*}\right) - (1 - \tau) \frac{\tilde{x}_1}{\bar{x}_*} \left(1 - \frac{\ell}{\bar{x}_*}\right) F'\left(\frac{\tilde{x}_1}{\bar{x}_*}\right) \leq -\frac{\tau r}{\lambda} < 0. \quad (\text{IB.55})$$

Thus, $\tilde{x}_1 > 0$, and

$$\begin{aligned} & \tau F\left(\frac{\tilde{x}_2}{\bar{x}_*}\right) - (1 - \tau) \frac{\tilde{x}_2}{\bar{x}_*} \left(1 - \frac{\ell}{\bar{x}_*}\right) F'\left(\frac{\tilde{x}_2}{\bar{x}_*}\right) \\ &= F\left(\frac{\tilde{x}_2}{\bar{x}_*}\right) \left[\tau - (1 - \tau) \left(1 - \frac{\ell}{\bar{x}_*}\right) \frac{\frac{\tilde{x}_2}{\bar{x}_*} F'\left(\frac{\tilde{x}_2}{\bar{x}_*}\right)}{F\left(\frac{\tilde{x}_2}{\bar{x}_*}\right)} \right] \\ &\leq F\left(\frac{\tilde{x}_2}{\bar{x}_*}\right) \left[\tau - (1 - \tau) \left(1 - \frac{\ell}{\bar{x}_*}\right) \frac{\frac{\tilde{x}_1}{\bar{x}_*} F'\left(\frac{\tilde{x}_1}{\bar{x}_*}\right)}{F\left(\frac{\tilde{x}_1}{\bar{x}_*}\right)} \right] \\ &= \frac{F\left(\frac{\tilde{x}_2}{\bar{x}_*}\right)}{F\left(\frac{\tilde{x}_1}{\bar{x}_*}\right)} \left[\tau F\left(\frac{\tilde{x}_1}{\bar{x}_*}\right) - (1 - \tau) \frac{\tilde{x}_1}{\bar{x}_*} \left(1 - \frac{\ell}{\bar{x}_*}\right) F'\left(\frac{\tilde{x}_1}{\bar{x}_*}\right) \right] < -\frac{\tau r}{\lambda}, \end{aligned} \quad (\text{IB.56})$$

where the first inequality uses Assumption 3, $\bar{x}_* \geq \tilde{x}_2 > \tilde{x}_1 > 0$, and $\bar{x}_* > \ell \geq 0$, and the second inequality uses (IB.55), $F'(z) > 0$ and $\tilde{x}_2 > \tilde{x}_1$. Comparing (IB.54) and (IB.56), we arrive at a contradiction.

Using (IB.50), we have

$$\mathcal{R}(\tilde{x}_0, \tilde{v}(\tilde{x}_0, \bar{x}_*)) = \mathcal{H}(\tilde{x}_0, \bar{x}_*) > \mathcal{R}(\tilde{x}_0, \bar{x}_*) \quad (\text{IB.57})$$

where the equality uses (IB.38). Since $\mathcal{R}_v(x, v) > 0$, we have $\tilde{v}(\tilde{x}_0, \bar{x}_*) > \bar{x}_*$. Using (IB.40), we can see that $\tilde{v}(\tilde{x}_0, \bar{x}) \leq \bar{x}$ for any $\bar{x} > \frac{1}{r-g}$. Thus, there exists $\bar{x}_1 > \bar{x}_*$, such that $\tilde{v}(\tilde{x}_0, \bar{x}_1) = \bar{x}_1$, which implies that

$$\bar{x}_1 = f_0(\tilde{x}_0/\bar{x}_1) \leq \bar{x}_*,$$

where the equality uses Proposition 6, and the inequality uses the definition of $\bar{x}_*$ given in (IB.14). Then we arrive again at a contradiction. ■

Proof of Proposition 7-(ii). According to (IB.51), when $c(\tilde{x}; \bar{x}) \leq 1$, we have⁵²

$$\mathcal{H}_{\tilde{x}}(\tilde{x}, \bar{x}) - \mathcal{R}_{\tilde{x}}(\tilde{x}, \bar{x}) = \tau r + \lambda \left[\tau F\left(\frac{\tilde{x}}{\bar{x}}\right) - (1 - \tau) \frac{\tilde{x}}{\bar{x}} \left(1 - \frac{\ell}{\bar{x}}\right) F'\left(\frac{\tilde{x}}{\bar{x}}\right) \right]. \quad (\text{IB.58})$$

Recall that (IB.12) holds for any $\tilde{x} \in \tilde{\mathcal{U}}$ and $\bar{x} \geq \tilde{x}$ satisfying (IB.3). Let

$$\xi(z) = \frac{\tau r}{\lambda} + \tau F(z) - (1 - \tau) z \left(1 - \frac{\ell}{f_0(z)}\right) F'(z). \quad (\text{IB.59})$$

Then we have $\xi(z) = [\mathcal{H}_{\tilde{x}}(z f_0(z), f_0(z)) - \mathcal{R}_{\tilde{x}}(z f_0(z), f_0(z))]/\lambda$ when $c(z f_0(z); f_0(z)) \leq 1$.

Step 1: We prove that one of the following cases holds:

- (i) $\xi(z) > 0$ for any $z \in [0, 1]$.
- (ii) $\xi(z) > 0$ for $z \in [0, 1)$, and $\xi(1) = 0$.
- (iii) There exists $z_1 \in (0, 1)$, such that $\xi(z_1) = 0$, $\xi(z) > 0$ for $z \in [0, z_1)$, and $\xi(z) < 0$ for $z \in (z_1, 1]$.

We derive from (IB.59) that

$$\begin{aligned} \frac{\ell(1 - \tau)}{f_0(z)} &= \frac{(1 - \tau) z F'(z) - \tau F(z) - (\tau r/\lambda - \xi(z))}{z F'(z)} \\ &= 1 - \tau - \frac{\tau/\lambda [r + \lambda F(z)] - \xi(z)}{z F'(z)}. \end{aligned} \quad (\text{IB.60})$$

When $c(\tilde{x}; \bar{x}) \leq 1$, using $c(\tilde{x}; \bar{x}) = r\tilde{x} + \lambda \int_0^{\tilde{x}/\bar{x}} (\tilde{x} - \ell z) dF(z) = \tilde{x}[r + \lambda F(\tilde{x}/\bar{x})] - \ell \lambda \int_0^{\tilde{x}/\bar{x}} z dF(z)$, we have $\theta(c(\tilde{x}; \bar{x})) = \tau - \tau c(\tilde{x}; \bar{x}) = \tau - \tau \left[\tilde{x}[r + \lambda F(\tilde{x}/\bar{x})] - \ell \lambda \int_0^{\tilde{x}/\bar{x}} z dF(z) \right]$. Substituting it into (IB.3), we obtain that

$$\begin{aligned} (r - g)\bar{x} &= 1 - \tau + \tau \left[\tilde{x}[r + \lambda F(\tilde{x}/\bar{x})] - \ell \lambda \int_0^{\tilde{x}/\bar{x}} Z dF(Z) \right] - \lambda(\bar{x} - \ell) \left(\int_0^{\tilde{x}/\bar{x}} Z dF(Z) \right) \\ &= (1 - \tau) \left[1 + \lambda \ell \left(\int_0^{\tilde{x}/\bar{x}} Z dF(Z) \right) \right] + \tau \tilde{x}[r + \lambda F(\tilde{x}/\bar{x})] - \lambda \bar{x} \left(\int_0^{\tilde{x}/\bar{x}} Z dF(Z) \right). \end{aligned}$$

⁵² When $c(\tilde{x}; \bar{x}) = 1$, we have $c(x; \bar{x}) < 1$ for $x < \tilde{x}$, and $\mathcal{H}_{\tilde{x}}(\tilde{x}, \bar{x}) - \mathcal{R}_{\tilde{x}}(\tilde{x}, \bar{x})$ refers to the left partial derivative $\mathcal{H}_{\tilde{x}}(\tilde{x}^-, \bar{x}) - \mathcal{R}_{\tilde{x}}(\tilde{x}^-, \bar{x})$.

Multiplying both sides of the above equation by $\ell/\bar{x}$ and using $z = \tilde{x}/\bar{x}$ and $\bar{x} = f_0(z)$, we obtain

$$(r - g)\ell = \frac{\ell(1 - \tau)}{f_0(z)} \left[1 + \lambda\ell \left(\int_0^z Z dF(Z) \right) \right] + \ell\tau z[r + \lambda F(z)] - \lambda\ell \int_0^z Z dF(Z). \quad (\text{IB.61})$$

Substituting (IB.60) into (IB.61), we obtain

$$\begin{aligned} & (r - g)\ell \\ &= \left[1 - \tau - \frac{\tau/\lambda[r + \lambda F(z)] - \xi(z)}{zF'(z)} \right] \left[1 + \lambda\ell \int_0^z Z dF(Z) \right] + \ell\tau z[r + \lambda F(z)] - \lambda\ell \int_0^z Z dF(Z) \\ &= 1 - \tau - \tau\lambda\ell \int_0^z Z dF(Z) - \frac{\tau/\lambda[r + \lambda F(z)] - \xi(z)}{zF'(z)} \left[1 + \lambda\ell \int_0^z Z dF(Z) \right] + \ell\tau z[r + \lambda F(z)] \\ &= 1 - \tau - \tau\lambda\ell \int_0^z Z dF(Z) + \tau[r + \lambda F(z)] \left[\ell z - \frac{1 + \lambda\ell \int_0^z Z dF(Z)}{\lambda z F'(z)} \right] + \xi(z) \frac{1 + \lambda\ell \int_0^z Z dF(Z)}{zF'(z)}. \end{aligned} \quad (\text{IB.62})$$

Let $\mathcal{V}(z) = -\lambda\ell \int_0^z Z dF(Z) + [r + \lambda F(z)] \left[\ell z - \frac{1 + \lambda\ell \int_0^z Z dF(Z)}{\lambda z F'(z)} \right]$. Then (IB.62) is equivalent to

$$(r - g)\ell - (1 - \tau) = \tau\mathcal{V}(z) + \xi(z) \frac{1 + \lambda\ell \int_0^z Z dF(Z)}{zF'(z)}, \quad (\text{IB.63})$$

where $(r - g)\ell - (1 - \tau) < 0$ since $\frac{1-\tau}{r-g} > \ell$ and $r > g$. According to Assumption 3, we have $(zF'(z)/F(z))' \geq 0$, for $z \in (0, 1)$. It follows that

$$F(z)(zF'(z))' - z(F'(z))^2 \geq 0, \quad z \in (0, 1). \quad (\text{IB.64})$$

For $z \in (0, 1]$, we have

$$\begin{aligned} & \mathcal{V}'(z) \\ &= -\lambda\ell zF'(z) + \lambda F'(z) \left[\ell z - \frac{1 + \lambda\ell \int_0^z Z dF(Z)}{\lambda z F'(z)} \right] + [r + \lambda F(z)] \frac{1 + \lambda\ell \int_0^z Z dF(Z)}{\lambda(zF'(z))^2} (zF'(z))' \\ &= \frac{1 + \lambda\ell \int_0^z Z dF(Z)}{\lambda(zF'(z))^2} \left[[r + \lambda F(z)](zF'(z))' - \lambda z(F'(z))^2 \right] \\ &= \frac{1 + \lambda\ell \int_0^z Z dF(Z)}{\lambda(zF'(z))^2} \left[r(zF'(z))' + \lambda \left(F(z)(zF'(z))' - z(F'(z))^2 \right) \right] > 0, \end{aligned} \quad (\text{IB.65})$$

where the inequality uses (IB.64) and $(zF'(z))' \geq z(F'(z))^2/F(z) > 0$ for $z > 0$. Since $zF'(z) \leq F(z)F'(1)/F(1) \rightarrow 0$ as $z \rightarrow 0$, we have $\lim_{z \rightarrow 0} \mathcal{V}(z) = -\infty$. When $\tau\mathcal{V}(1) < (r - g)\ell - (1 - \tau)$, we have $\tau\mathcal{V}(z) < (r - g)\ell - (1 - \tau)$ for any $z \in [0, 1]$. Substituting it into (IB.63), we have $\xi(z) > 0$ for any $z \in [0, 1]$. Thus the result in case (i) holds.

Next, we consider the case $\tau\mathcal{V}(1) \geq (r - g)\ell - (1 - \tau)$. Then the following equation admits a unique solution in the region $z \in [0, 1]$, denoted by z_1 .

$$(r - g)\ell - (1 - \tau) = \tau\mathcal{V}(z). \quad (\text{IB.66})$$

We have $z_1 \in (0, 1]$ as $\lim_{z \rightarrow 0} \mathcal{V}(z) = -\infty$. As $\mathcal{V}(z)$ is increasing in z and $(r-g)\ell - (1-\tau) = \tau\mathcal{V}(z_1)$, we have $(r-g)\ell - (1-\tau) > \tau\mathcal{V}(z)$ for $z \in [0, z_1)$, and $(r-g)\ell - (1-\tau) < \tau\mathcal{V}(z)$ for $z \in (z_1, 1]$.

When $z_1 = 1$, we derive from (IB.63) that $\xi(z) > 0$ for $z \in [0, 1)$ and $\xi(1) = 0$. Then the result in case (ii) holds.

When $z_1 \in (0, 1)$, we derive from (IB.63) that $\xi(z) > 0$ for $z \in [0, z_1)$, and $\xi(z) < 0$ for $z \in (z_1, 1]$. Then the result in case (iii) holds.

Step 2: We prove that for the pair $(\tilde{x}, \bar{x})$ satisfying $\bar{x} \geq \tilde{x} \geq 0$, $\bar{x} = \mathcal{A}(\bar{x}) = \tilde{v}(\tilde{x}, \bar{x})$, we have $\tilde{x} > 0$,

$$c(\tilde{x}; \bar{x}) \leq 1, \quad \text{and} \quad \mathcal{H}_{\tilde{x}}(\tilde{x}, \bar{x}) - \mathcal{R}_{\tilde{x}}(\tilde{x}, \bar{x}) \geq 0, \quad (\text{IB.67})$$

where $\mathcal{A}(\cdot)$ is given in (IB.9).

We have $\bar{x} > \ell \geq 0$ using (IB.46). We first prove that $\tilde{x} > 0$. For the sake of contradiction, assume $\tilde{x} = 0$. According to (IB.51), we have $\mathcal{H}_{\tilde{x}}(0, \bar{x}) - \mathcal{R}_{\tilde{x}}(0, \bar{x}) = \tau r > 0$. Then there exists $\varepsilon > 0$, such that

$$\mathcal{H}(\varepsilon, \bar{x}) - \mathcal{R}(\varepsilon, \bar{x}) > \mathcal{H}(0, \bar{x}) - \mathcal{R}(0, \bar{x}) = 0, \quad (\text{IB.68})$$

where the equality uses $\bar{x} = \tilde{v}(0, \bar{x})$. Since $\mathcal{R}(\varepsilon, \tilde{v}(\varepsilon, \bar{x})) = \mathcal{H}(\varepsilon, \bar{x}) > \mathcal{R}(\varepsilon, \bar{x})$ and $\mathcal{R}_v > 0$, we have $\tilde{v}(\varepsilon, \bar{x}) > \bar{x}$. Then we arrive at a contradiction. Thus, $\tilde{x} > 0$.

Next, we prove that $\mathcal{H}_{\tilde{x}}(\tilde{x}, \bar{x}) - \mathcal{R}_{\tilde{x}}(\tilde{x}, \bar{x}) \geq 0$. For the sake of contradiction, assume $\mathcal{H}_{\tilde{x}}(\tilde{x}, \bar{x}) - \mathcal{R}_{\tilde{x}}(\tilde{x}, \bar{x}) < 0$. Then there exists $x_1 \in (0, \tilde{x})$, such that

$$\mathcal{H}(x_1, \bar{x}) - \mathcal{R}(x_1, \bar{x}) > \mathcal{H}(\tilde{x}, \bar{x}) - \mathcal{R}(\tilde{x}, \bar{x}) = 0, \quad (\text{IB.69})$$

where the equality uses $\bar{x} = \tilde{v}(\tilde{x}, \bar{x})$. Since $\mathcal{R}(x_1, \tilde{v}(x_1, \bar{x})) = \mathcal{H}(x_1, \bar{x}) > \mathcal{R}(x_1, \bar{x})$ and $\mathcal{R}_v > 0$, we have $\tilde{v}(x_1, \bar{x}) > \bar{x}$. Then we arrive at a contradiction. Thus, $\mathcal{H}_{\tilde{x}}(\tilde{x}, \bar{x}) - \mathcal{R}_{\tilde{x}}(\tilde{x}, \bar{x}) \geq 0$.

According to (IB.51), we have

$$\mathcal{H}_{\tilde{x}}(\tilde{x}, \bar{x}) - \mathcal{R}_{\tilde{x}}(\tilde{x}, \bar{x}) = -\lambda \frac{\tilde{x}}{\bar{x}} \left(1 - \frac{\ell}{\bar{x}}\right) F' \left(\frac{\tilde{x}}{\bar{x}}\right) < 0, \quad \text{if} \quad c(\tilde{x}; \bar{x}) > 1. \quad (\text{IB.70})$$

Since we have shown that $\mathcal{H}_{\tilde{x}}(\tilde{x}, \bar{x}) - \mathcal{R}_{\tilde{x}}(\tilde{x}, \bar{x}) \geq 0$, we can derive from (IB.70) that $c(\tilde{x}, \bar{x}) \leq 1$, which proves (IB.67).

Step 3: We prove that for any pair $(\tilde{x}, \bar{x})$ satisfying $\bar{x} > \tilde{x} \geq 0$, $\bar{x} = \mathcal{A}(\bar{x}) = \tilde{v}(\tilde{x}, \bar{x})$, and $c(\tilde{x}; \bar{x}) < 1$, we have

$$\mathcal{H}_{\tilde{x}}(\tilde{x}, \bar{x}) - \mathcal{R}_{\tilde{x}}(\tilde{x}, \bar{x}) = 0, \quad (\text{IB.71})$$

where $\mathcal{A}(\cdot)$ is given in (IB.9).

For the sake of contradiction, assume that $\mathcal{H}_{\tilde{x}}(\tilde{x}, \bar{x}) - \mathcal{R}_{\tilde{x}}(\tilde{x}, \bar{x}) \neq 0$. Using $c(\tilde{x}; \bar{x}) < 1$ and (IB.51), we can see that there exists a small $\varepsilon > 0$, such that $\tilde{x} + \varepsilon < \bar{x}$, and $\mathcal{H}_{\tilde{x}}(x, \bar{x}) - \mathcal{R}_{\tilde{x}}(x, \bar{x})$ is continuous in $x \in (\tilde{x} - \varepsilon, \tilde{x} + \varepsilon)$. It follows that there exists $x \in (\tilde{x} - \varepsilon, \tilde{x} + \varepsilon)$, such that

$$\mathcal{H}(x, \bar{x}) - \mathcal{R}(x, \bar{x}) > \mathcal{H}(\tilde{x}, \bar{x}) - \mathcal{R}(\tilde{x}, \bar{x}) = 0, \quad (\text{IB.72})$$

where the equality uses $\bar{x} = \tilde{v}(\tilde{x}, \bar{x})$. Since $\mathcal{R}_v > 0$, we have $\mathcal{H}(x, \bar{x}) > \mathcal{R}(x, \bar{x}) \geq \mathcal{R}(x, x)$, so that $x \in \mathbb{U}(\bar{x})$ using (IB.37). Since $\mathcal{R}(x, \tilde{v}(x, \bar{x})) = \mathcal{H}(x, \bar{x}) > \mathcal{R}(x, \bar{x})$ and $\mathcal{R}_v > 0$, we have $\tilde{v}(x, \bar{x}) > \bar{x} = \mathcal{A}(\bar{x})$. Recalling that $x \in \mathbb{U}(\bar{x})$ and (IB.9), we arrive at a contradiction. This proves (IB.71).

Step 4: We prove that there exists a unique $\bar{x}$ such that $\mathcal{A}(\bar{x}) = \bar{x}$.

For the sake of contradiction, assume there exist $\bar{x}_1 \neq \bar{x}_2$, such that $\mathcal{A}(\bar{x}_i) = \bar{x}_i$, $i = 1, 2$. Then, there exists $\tilde{x}_i \in \tilde{\mathbb{U}}$, such that $\bar{x}_i = \tilde{v}(\tilde{x}_i, \bar{x}_i) \geq \tilde{v}(\tilde{x}, \bar{x}_i)$ for any $\tilde{x} \in \mathbb{U}(\bar{x}_i)$. According to (IB.12) and the result proved in step 2, there exist $0 < z_1 < z_2 \leq 1$, such that $\tilde{x}_1 = z_1 f_0(z_1)$, $\bar{x}_1 = f_0(z_1)$, $\tilde{x}_2 = z_2 f_0(z_2)$, $\bar{x}_2 = f_0(z_2)$. According to the result proved in step 2, we have $c(\tilde{x}_i; \bar{x}_i) \leq 1$, $i = 1, 2$. Combining it with the result proved in step 2 of the proof of Proposition 6, we can see that

$$c(z f_0(z); f_0(z)) < 1, \quad z \in [0, z_2]. \quad (\text{IB.73})$$

In particular, $c(z_1 f_0(z_1); f_0(z_1)) < 1$, which together with $z_1 < 1$ and the result proved in step 3 implies that $\mathcal{H}_{\tilde{x}}(z_1 f_0(z_1), f_0(z_1)) - \mathcal{R}_{\tilde{x}}(z_1 f_0(z_1), f_0(z_1)) = 0$. Combining it with (IB.58) and (IB.59), we have $\xi(z_1) = 0$. Then we derive from the result proved in step 1 that $\xi(z) < 0$ for any $z \in (z_1, 1]$. Using $\tilde{x}_2 = z_2 f_0(z_2)$, $\bar{x}_2 = f_0(z_2)$ and the result proved in step 2, we conclude that $\mathcal{H}_{\tilde{x}}(z_2 f_0(z_2); f_0(z_2)) - \mathcal{R}_{\tilde{x}}(z_2 f_0(z_2); f_0(z_2)) \geq 0$, and thus $\xi(z_2) \geq 0$. Thus we arrive at a contradiction.

According to the proof of Theorem 5, we know that the set of default boundaries in all equilibria $\in \mathcal{S}_x$ can be represented by (IB.44). In step 4 above, we have proved the uniqueness of $\bar{x}$, and thus $\bar{x} = \bar{x}_*$, which is defined in (IB.14). In Proposition 7-(i), we have proved the uniqueness of the target debt-EBIT ratio achieving the highest equilibrium value $\bar{x}_*$. This proves the uniqueness of equilibrium. ■

IC Accounting Covenants: Market Leverage and Coverage Ratios

Market leverage (ML) is the value of debt divided by the total firm value. It is the widely used loan-to-value (LTV) ratio in real estate finance. In our model, $ML_t = x_t/v(x_t)$ where $v(x_t) = x_t + p(x_t)$. A common liquidity metric used by practitioners to measure a firm's ability to service its debt is the *interest coverage ratio*, which we denote by ICR_t :

$$ICR_t = \frac{Y_t}{C_t}. \quad (\text{IC.1})$$

The ICR_t measures the firm's ability to service its debt interest payments out of current EBIT. In our model, $ICR_t = 1/c(x_t)$. We can also evaluate the firm's ability to meet the sum of interest and tax payments with its contemporaneous EBIT with the following *liquidity coverage ratio*:

$$LCR_t = \frac{Y_t}{C_t + \Theta_t}. \quad (\text{IC.2})$$

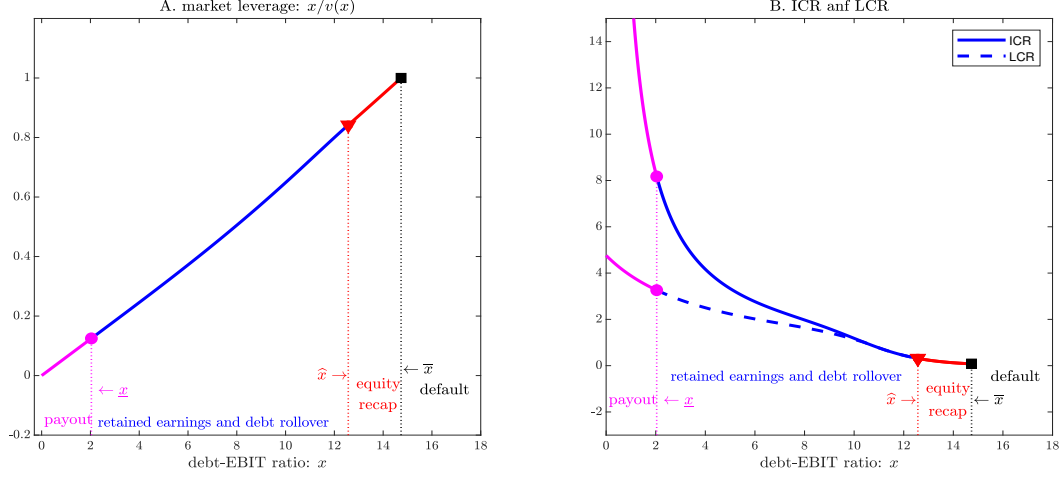


Figure 20: Market leverage ML , interest coverage ratio ICR , and liquidity coverage ratio LCR .

When $LCR_t > 1$, the firm's earnings are positive, so in the earnings-retention region (where the firm neither issues equity nor pays dividends) its debt balance X_t automatically decreases.⁵³ In our model, $LCR_t = 1/[c(x_t) + \theta(c(x_t))]$. These ratios are widely used in debt covenants.⁵⁴

Figure 20 plots market leverage ML , the interest coverage ratio ICR and the liquidity coverage ratio LCR as functions of x under the baseline calibration. As expected, market leverage increases with the debt-EBIT ratio x , whereas both coverage ratios, ICR and LCR , increase significantly as x decreases.

ID Solution for the Investment and Growth Option Model in Appendix J

In this appendix, we first solve the extended model with growth option and capital accumulation in Appendix J and report parameter choices for the quantitative analysis used in that section.

⁵³ Strebulaev (2007) defines financial distress as situations where the firm's ICR_t falls below one and requires the firm to take corrective actions (e.g., inefficient asset sales) to alleviate debt burden under financial distress. We share his view that liquidity measures such as ICR and LCR are critically important for capital structure decisions. However, in Strebulaev (2007) the firm continuously issues equity or makes dividend payments to shareholders and there is no earnings retention. Leverage dynamics in his model are fundamentally different from ours in which earnings retention, debt rollover, equity issuance, and payout jointly determine leverage dynamics.

⁵⁴ It is straightforward to introduce other covenants such as cash-flow-based covenants that limit the firm's borrowing. Lian and Ma (2021) document that 80% of debt (for US nonfinancial firms) is backed predominantly by cash flows from operations (cash flow/earnings based lending). Our model is well suited to capture these empirically important cash flow based lending activities.

ID.1 Equity Value in Mature Phases: $s = 1$ and $s = 2$

The firm must be in one of the two mature phases after the arrival of the growth option at time T^G . If it has exercised the growth option at T^G , the firm is in phase 2 with productivity A_2 . Otherwise, it is in phase 1 with productivity $A_1 < A_2$. The only difference between the two mature phases is productivity. In phase $s = 1, 2$, the firm's equity value $P(K, X; A_s)$ satisfies the following HJB equation in the earnings-retention and debt-rollover region:

$$\begin{aligned} rP(K, X; A_s) = & \max_I [C + \Theta(C, K) - (A_s K - I)] P_X + \Psi(I, K) P_K + \frac{\sigma_K^2 K^2}{2} P_{KK} \\ & + \lambda_K \mathbb{E}[P(ZK, X; A_s) - P(K, X; A_s)] , \end{aligned} \quad (\text{ID.1})$$

where $\Theta = \Theta(C, K)$ is the tax payment. Let $c_t = C_t/K_t$ and $x_t = X_t/K_t$. As in the baseline model of Section 2, $c(x) = (r + \eta(x))x$ and the equilibrium credit spread $\eta(x)$ is given by

$$\eta(x) = \lambda_K \int_0^{x/\bar{x}} \left(1 - \frac{\ell_K z}{x} \right) dF(z) . \quad (\text{ID.2})$$

We write $\Theta(C_t, K_t) = \theta(c_t)K_t$, where $\theta(c) = \tau(A_s - c)\mathbf{1}_{c \leq A_s}$.

Using the homogeneity property in K , we obtain the following ODE for scaled equity value $p_s(x) = P(K, X; A_s)/K$ in the earnings-retention and debt-rollover region where $\underline{x}_s \leq x < \hat{x}_s$ for $s = 1$ and $s = 2$:

$$\begin{aligned} rp_s(x) = & \max_i -[A_s - i - c(x) - \theta(c(x))] p'_s(x) + \psi(i)(p_s(x) - xp'_s(x)) \\ & + \frac{\sigma_K^2 x^2}{2} p''_s(x) + \lambda_K \mathbb{E}[Zp_s(x/Z) - p_s(x)] . \end{aligned} \quad (\text{ID.3})$$

ID.2 Equity Value in the Growth Phase: $s = 0$

In the growth phase ($s = 0$), the firm's equity value $P(K, X; A_0)$ satisfies the following HJB equation in the earnings-retention and debt-rollover region:

$$\begin{aligned} rP(K, X; A_0) = & \max_{I, \mathbf{1}^A} [C + \Theta - (A_0 K - I)] P_X + \Psi(I, K) P_K + \frac{\sigma_K^2 K^2}{2} P_{KK} \\ & + \lambda_K \mathbb{E}[P(ZK, X; A_0) - P(K, X; A_0)] \\ & + \lambda_G [\mathbf{1}^A P(K, X + \phi K; A_2) + (1 - \mathbf{1}^A) P(K, X; A_1) - P(K, X; A_0)] , \end{aligned} \quad (\text{ID.4})$$

The indicator function $\mathbf{1}^A$ is a control variable that equals one if and only if the firm exercises its growth option at the arrival time T^G . If it exercises the option, the firm incurs a one-time cost of ϕK and its productivity permanently increases to A_2 . Otherwise, the firm enters into the mature phase with productivity A_1 .

Using the homogeneity property, we obtain the following ODE for scaled equity value $p_0(x) = P(K, X; A_0)/K$ in the earnings-retention and debt-rollover region where $\underline{x}_0 \leq x < \widehat{x}_0$:

$$\begin{aligned} rp_0(x) = \max_{i, \mathbf{1}^A} & -[A_0 - i - c(x) - \theta(c(x))]p'_0(x) + \psi(i)(p_0(x) - xp'_0(x)) \\ & + \frac{\sigma_K^2 x^2}{2} p''_0(x) + \lambda_K \mathbb{E}[Zp_0(x/Z) - p_0(x)] \\ & + \lambda_G [\mathbf{1}^A p_2(x + \phi) + (1 - \mathbf{1}^A)p_1(x) - p_0(x)] . \end{aligned} \quad (\text{ID.5})$$

Equation (ID.5) implies that it is optimal to exercise the growth option when the firm obtains a higher equity value $p_2(x + \phi) \geq p_1(x)$. The firm pits the benefit of a higher productivity A_2 going forward against the cost of exercising the growth option: ϕ per unit of capital K .⁵⁵

ID.3 Investment in Phase $s = 0, 1, 2$

Since the firm immediately leaves the equity-issuance region and the payout region, it chooses investment I only in the earnings-retention and debt-rollover region. In all three phases $s = 0, 1, 2$, investment I is determined by the FOC:

$$\frac{1}{\psi'(i_s(x))} = -\frac{q_s^m(x)}{p'_s(x)}, \quad (\text{ID.6})$$

where $-p'_s(x) > 1$ is shareholders' marginal cost of servicing debt, $q_s^m(x)$ is the firm's *marginal* q given by $q_s^m = V_K(K, X; A_s) = p_s(x) - xp'_s(x)$. That is, investment is determined by the ratio between marginal q and shareholders' marginal cost of debt: $-p'_s(x) = 1 - v'(x)$. Differentiating (ID.6) and using the concavity of $\psi(\cdot)$, we see that investment is decreasing in x when $p''(x) < 0$ and increasing in x when $p''(x) > 0$.⁵⁶

ID.4 Other Conditions in the Growth and Mature Phases ($s = 0, 1, 2$)

In the payout region the following condition holds:

$$p_s(x) = p_s(\underline{x}_s) + \underline{x}_s - x, \quad \text{for } x < \underline{x}_s, \quad (\text{ID.7})$$

and the payout boundary $\underline{x}_s$ satisfies

$$p'_s(\underline{x}_s) = -1, \quad \text{and} \quad p''_s(\underline{x}_s) = 0. \quad (\text{ID.8})$$

⁵⁵ If $x + \phi$ exceeds $\widehat{x}_2$ and if $p_2(x + \phi) \geq p_1(x)$, the firm recapitalizes to $\tilde{x}_2$ after exercising the option.

⁵⁶ Dropping the subscript for phase s to simplify notations, we obtain:

$$i'(x) = -\frac{(\psi'(i(x)))^2}{\psi''(i(x))} \frac{\partial}{\partial x} \left[\frac{q_m(x)}{-p'(x)} \right] = -\underbrace{\frac{(\psi'(i(x)))^2}{\psi''(i(x))} \frac{p(x)}{(p'(x))^2}}_{\text{positive as } \psi(\cdot) \text{ is concave}} p''(x),$$

which implies that the sign of $i'(x)$ is the same as that of $p''(x) = v''(x)$, as $v(x) = p(x) + x$.

In the equity-issuance region, equity value satisfies:

$$p_s(x) = p_s(\tilde{x}_s) - [h_0^K + (1 + h_1)(x - \tilde{x}_s)] \quad , \quad \hat{x}_s \leq x \leq \bar{x}_s \quad , \quad (\text{ID.9})$$

where $\tilde{x}_s$ is the equity recapitalization target in phase s and

$$-p'_s(x) = -p'_s(\tilde{x}_s) = 1 + h_1 \quad , \quad \hat{x}_s \leq x < \bar{x}_s \quad . \quad (\text{ID.10})$$

Finally the firm defaults and $p_s(x) = 0$ for $x > \bar{x}_s$.

ID.5 Summary

In phase s , equity value $p_s(x)$ takes different forms: i) in the payout region ($x < \underline{x}_s$) where $p_s(x)$ is linear in x as given in (ID.7); ii) in the earnings-retention and debt-rollover region ($\underline{x}_s \leq x < \hat{x}_s$) where $p_s(x)$ satisfies the integro-differential equation (ID.3) for $s = 1, 2$ and $p_0(x)$ satisfies the integro-differential equation (ID.5); iii) in the equity-issuance region ($\hat{x}_s \leq x \leq \bar{x}_s$) where $p_s(x)$ is given by (ID.9); and iv) in the default region ($x > \bar{x}_s$) where $p_s(x) = 0$. The equilibrium credit spread is given by (ID.2). In the growth phase, the firm exercises its growth option if and only if $p_2(x + \phi) \geq p_1(x)$. In all phases the optimal investment policy satisfies (ID.6), the optimal target leverage (payout threshold) $\underline{x}_s$ satisfies (ID.8), both the recapitalization target $\tilde{x}_s$ and the equity-issuance boundary satisfy (ID.10), and the debt capacity $\bar{x}_s$ is the lowest value of x where $p_s(x) = 0$.

ID.6 Parameter Choices

We use a quadratic function: $\psi(i) = i - \frac{\xi}{2}i^2 - \delta_K$ where $\xi > 0$ measures how costly it is to adjust the capital stock and δ_K is the depreciation rate. As in our EBIT-based model of Section 2, the distribution of $Z \in [0, 1]$ is assumed to be given by $F(z) = z^{\beta_K}$, where $\beta_K > 0$. To ease comparison with the EBIT-based model, we use $r = 6\%$, $\lambda_K = 1.2$, $\beta_K = 9$, $\delta_K = 6\%$, $\sigma_K = 40.6\%$, $h_0^K = 0.01$ and $h_1 = 0.09$. As for the remaining parameters, in the mature phase without exercising the growth option we set $\xi = 1.82$, $A_1 = 34.1\%$, and $\ell_K = 0.4$ so as to target: i) an average investment-capital ratio i of 10%; ii) a mean average q of 1.24; and iii) a debt recovery upon default of 51% of the debt face value. These parameter values are in line with the values used in dynamic corporate finance and q theory literatures. The quantitative predictions of this extended model can be related to the findings in Lian and Ma (2021), especially their comparison of asset-value-based versus cash-flow-based borrowing constraints.

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