

Option Exercise Games and a Duopolistic q Theory of Investment*

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Abstract

We address Back and Paulsen’s questions of how to define closed-loop equilibrium in a classic real-option game and whether such an equilibrium can sustain positive profits. Our definition combines a marginal q condition ensuring equilibria are impervious to preemption with the Back–Paulsen zero-profit equilibrium as punishment. We characterize infinitely many such equilibria in closed form and develop a duopolistic q theory of irreversible investment. Investment depends on marginal q and cross marginal q , the sensitivity of a firm’s value to its rival’s capital stock, and proceeds more slowly than in the Back–Paulsen equilibrium but faster than in the open-loop equilibrium.

Keywords: real-option games, closed-loop equilibria, irreversible investment, duopoly, marginal q , cross marginal q

JEL Codes: C61, C73, D43, E22, L13

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1 Introduction

How do firms exercise their growth options and accumulate capital under uncertainty in an oligopolistic environment? Grenadier (2002) develops a real-options game framework in which firms undertake irreversible investment in a setting where the output price decreases with industry capital stock. Back and Paulsen (2009) show that Grenadier's solution is an open-loop equilibrium. In this equilibrium, each firm chooses its investment strategy at time 0 and commits to it regardless of its competitor's actions, even though the dynamic game provides no mechanism to make such commitment credible. In particular, if one firm deviates from its prescribed strategy by investing more, its competitor is required to ignore the deviation and continue following its original investment plan. However, such behavior is inconsistent with subgame perfection.

In light of this observation, Back and Paulsen (2009) argue that a closed-loop equilibrium is a more appropriate solution concept, as it allows firms to condition their strategies at time t on the history of the game up to t and thus respond dynamically to competitors' actions over time, thereby yielding a subgame-perfect equilibrium. Back and Paulsen (2009) also identify a closed-loop equilibrium in which firms optimally follow the simple net present value (NPV) rule for investment. Notably, this closed-loop equilibrium yields zero economic profits and completely erodes the value of industry growth options.

Furthermore, Back and Paulsen (2009) highlight several open questions and avenues for future research. In their introduction, they note: *"There are difficulties in even defining the game in closed-loop form."* They further conjecture the existence of alternative equilibria in which firms earn positive profits and preserve some option value of waiting, but state that *"It is an open question whether the perfectly competitive boundary is the unique boundary such that closed-loop strategies ... are impervious to preemption."*

We respond to these open questions by first defining a closed-loop equilibrium and then identifying a continuum of such equilibria in which firms earn positive profits. Using an explicit solution to a closed-loop equilibrium, we further develop a tractable duopolistic q -theory of investment and show that, despite competition, a substantial fraction of the option value is preserved in equilibrium.

To set the stage, Section 2 introduces a widely studied duopoly real-options investment game (Abel and Eberly, 1996; Grenadier, 2002), in which two firms face an exogenous demand shock process $\{X_t\}$ and a downward-sloping inverse demand function, and compete by making recurrent irreversible investments.¹ In Section 3, we summarize the monopoly solution obtained by Abel and Eberly (1996) and then review Back and Paulsen’s definition of open-loop equilibrium, together with the associated solution derived by Grenadier (2002) and formalized in Back and Paulsen (2009). The monopoly and open-loop equilibrium solutions are two useful benchmarks against which we compare our closed-loop equilibrium solution.

Section 4 is at the core of our analysis. In this section, we define closed-loop equilibria in which firms earn positive profits for the option exercise games, inspired by Back and Paulsen’s conjecture that *“there should be equilibria . . . , in which firms invest less than the perfectly competitive amount and threaten to punish any firm that deviates.”* To define a closed-loop equilibrium with positive profits, we impose the following two conditions.

The first is the *marginal q condition*, which requires that at any time t a firm’s marginal benefit from investment not exceed its capital purchase cost. It requires that no firm can gain by a lumpy (preemptive) investment even if its rival does not punish the deviation; in this sense our equilibria are impervious to preemption, the property that Back and Paulsen ask for. The second condition is a grim-trigger condition that specifies the consequences of a one-shot deviation from the equilibrium strategy. Fix a small time interval ε during which a firm deviates. Its competitor observes the deviation at the end of this interval and responds by switching to the leader’s strategy in Back and Paulsen’s zero-profit closed-loop equilibrium. This grim-trigger strategy deters such deviations.

To complete the definition of a closed-loop equilibrium, we characterize a firm’s feedback strategy using the running maximum process $\{M_t\}$ associated with the exogenous demand shock process $\{X_t\}$. While it is known that, in recurrent real-options investment settings, a firm increases its capital stock if and only if the running maximum process $\{M_t\}$ reaches a new high (see, e.g., Abel and Eberly, 1996), we are the first to use the $\{M_t\}$ process to characterize feedback strategies in a duopolistic closed-loop equilibrium.

¹While we focus on the case with a linear inverse demand function, our analysis also applies to other settings, e.g., when inverse demand function is isoelastic (see Internet Appendix IA.C).

Having defined a closed-loop equilibrium in Section 4, we develop a solution method to identify and characterize symmetric closed-loop equilibria in Section 5. We show that there exist infinitely many economically intuitive closed-loop equilibria in which firms earn positive profits. A key part of our analysis is to characterize the endogenous boundary that determines a firm's investment and inaction regions. Given this boundary, $\mathcal{X}$, which depends on the pair of capital stocks (K_t^a, K_t^b) , the state space is divided into four distinct regions. When industry demand $X_t < \mathcal{X}(K_t^a, K_t^b)$, neither firm invests, as the industry has excess production capacity; that is, the aggregate capital stock is too large. When industry demand $X_t > \mathcal{X}(K_t^a, K_t^b)$, the larger firm remains inactive while the smaller firm invests until the threshold $X_t = \mathcal{X}(K_t^a, K_t^b)$ is reached. This is because the smaller firm has a higher marginal revenue from investment. When $X_t > \mathcal{X}(K_t^a, K_t^b)$ and the two firms are equal in size, both invest simultaneously to maintain equality along the 45° line $K_t^a = K_t^b$ until the threshold $X_t = \mathcal{X}(K_t^a, K_t^b)$ is reached.

But how is the equilibrium investment boundary $\mathcal{X}$ determined? In a given subgame of a closed-loop equilibrium, the following two conditions must hold: (i) when a firm invests, its marginal q equals the capital purchase cost p ; and (ii) when its competitor invests, the firm's value must be locally insensitive to the competitor's capital stock, so that its cross-marginal q equals zero. These two conditions imply the following boundary conditions on the endogenous investment boundary $\mathcal{X}(K_t^a, K_t^b)$ for firm a : (1) marginal q equals p on the segment of $\mathcal{X}(K_t^a, K_t^b)$ with $K_t^a \leq K_t^b$, where firm a invests; and (2) cross-marginal q equals zero on the segment of $\mathcal{X}(K_t^a, K_t^b)$ with $K_t^a \geq K_t^b$, where the competitor, firm b , invests. However, as the investment boundary $\mathcal{X}$ is endogenous, we need one more condition.

One natural conjecture would be to impose a smooth-pasting condition, whose economic intuition is illustrated by Dixit (1993) in various applications. However, unlike in a single firm's irreversible investment problem—see, e.g., Abel and Eberly (1996)—this condition does not hold in any closed-loop equilibrium, including Back and Paulsen's zero-profit closed-loop equilibrium. In a single-firm problem, the investment boundary is determined solely by the firm's local indifference between waiting and investing, resulting in a smooth-pasting condition. By contrast, in a closed-loop equilibrium, the investment boundary must account for a firm's dynamic response to its competitor in every subgame. Therefore, the boundary is determined by subgame perfection rather than by a tangency condition arising from a single-

firm optimization problem. Consequently, marginal q reaches the investment cost p without smoothly pasting to it.²

Instead of imposing an invalid smooth-pasting condition, we conjecture and verify the existence of a class of closed-loop equilibria in which the investment threshold $\mathcal{X}(K_t^a, K_t^b)$ is *piecewise linear* in the two firms' capital stocks. This conjecture is inspired by the functional forms of the investment boundaries in Abel and Eberly (1996)'s single-firm problem and Grenadier (2002)'s open-loop equilibrium. In Section 6, we show that the admissible set of piecewise linear trigger strategies is characterized by a pair of parameters, (θ_1, θ_2) , and satisfies the two inequalities implied by the equilibrium conditions introduced in Section 4: one corresponding to the marginal q condition and the other to the grim-trigger strategy.

Given the important roles of jumps in our model, we make the following observations. First, while a firm is allowed to jump its capital stock at any time, our equilibrium conditions ensure that using a jump to preempt is an unprofitable deviation. Second, implementing the grim-trigger strategy as a punishment calls for a jump to the BP zero-profit closed-loop equilibrium. These two observations together imply that along the equilibrium path a firm may jump its investment only at time zero in our singular control setting.

Next, in Section 7 we use our closed-loop equilibrium solution to develop a tractable duopolistic q theory of investment. In all newly identified closed-loop equilibria, firms earn positive profits by preserving some option value of waiting, reflecting less aggressive competition than in Back and Paulsen's closed-loop equilibrium. Compared to the open-loop equilibrium, firms invest earlier, resulting in lower firm values across all closed-loop equilibria. Using demand processes and parameter values commonly used in the real-option game literature, our calculations show that the closed-loop equilibrium with the highest industry value features investment rates 10% faster than in the open-loop equilibrium and 26% slower than in Back and Paulsen's closed-loop equilibrium. Under the same baseline parameters, while varying volatility, we find that a duopoly industry preserves between 61% and 89% of the maximal (monopoly) industry value attained under collusion (Abel and Eberly, 1996).³ By

²The smooth-pasting condition also fails to hold in Back and Paulsen's zero-profit closed-loop equilibrium for the same reason. By contrast, Grenadier (2002) and Back and Paulsen (2009) use smooth-pasting conditions to characterize open-loop equilibria, which are defined under time-0 commitment and therefore do not require subgame perfection.

³We obtain these results by increasing σ from 0.05 toward 0.2, while fixing $r = 6\%$ and $\mu = 1\%$ per annum.

contrast, in the open-loop equilibrium, the industry preserves 8/9 of the monopoly industry value. In summary, although closed-loop competition erodes some option value relative to the open-loop equilibrium, a significant fraction of that value is still preserved under imperfect competition.⁴

In Section 8, we show that in addition to closed-loop equilibria with piecewise linear triggers, there also exist closed-loop equilibria with *nonlinear* investment triggers. Compared to the closed-loop equilibria with piecewise linear investment triggers analyzed in Section 6, firms compete less aggressively when market demand is low but more fiercely as market demand increases due to the nonlinearity of the investment trigger. Economically, in these equilibria with nonlinear investment triggers, the larger firm’s marginal q also equals the marginal cost of purchasing capital when the smaller firm invests. We conclude in Section 9 and provide supporting technical details in Appendices A-B and Internet Appendices.

Related literature. As Back and Paulsen (2009) noted, “the distinction between open-loop and closed-loop (or feedback) strategies is well understood in the context of *deterministic* oligopoly investment games. See Fershtman and Muller (1984), Reynolds (1987), Tirole (1988), or Fudenberg and Tirole (1992). Equilibria in open-loop strategies are unattractive because they fail subgame or Markov perfection.” However, introducing shocks into an irreversible investment model fundamentally changes the analysis, making even the definition of a closed-loop equilibrium for the game conceptually challenging, let alone an explicit characterization of a closed-loop equilibrium, as emphasized by Back and Paulsen (2009).

A key contribution of our work is to provide a closed-loop equilibrium definition and characterize a class of infinitely many closed-loop equilibria in closed form. Central to our closed-loop equilibrium analysis is the running maximum process associated with the demand shock, $\{M_t\}$, as a state variable. This idea is inspired by a monopoly’s irreversible investment problem where it is optimal to invest whenever the running maximum reaches a new high.

Note that this calculation only depends on the interest rate r , demand growth μ , and demand volatility σ .

⁴An alternative approach to modeling investment frictions is to use convex capital adjustment costs. Because convexity ensures interior solutions, a firm’s optimal investment—even in a duopolistic setting—can then be characterized by a first-order condition, thereby avoiding the difficulties posed by singular control in our analysis. This tractability, however, comes at a cost: it shuts down the option-value-of-waiting channel, which we have shown to remain quantitatively significant despite competition.

While the mathematical formulation of a monopoly’s irreversible investment problem is well understood, how to formulate a firm’s subgame best response to its competitor’s strategy in an irreversible investment duopoly game has not been worked out in the literature. We fill in this gap by formulating a firm’s irreversible investment as a response to an increase of $\{M_t\}$. A key is to specify the response magnitude as a function of the state vector $Z_t = (X_t, K_t^a, K_t^b)$. This allows us to recursively characterize strategies by working with $\{Z_t\}$ and the running maximum $\{M_t\}$.

This recursive formulation allows us to connect to and build on the differential game literature, see, for instance, Fershtman and Muller (1984), Reynolds (1987), and Fudenberg and Tirole (1992). In a standard differential game, firm i chooses an investment at a rate of u_t^i , so that its capital stock evolves according to $dK_t^i = u_t^i dt$. By contrast, in our setting, firm i ’s investment follows $dK_t^i = u_t^i dM_t$, where $u_t^i = \mathbf{u}^i(Z_t)$ is the time- t investment rate in response to an increase of M_t .⁵ In summary, our formulation can thus be viewed as a feedback differential game that operates on the running maximum $\{M_t\}$, capturing the singular nature of irreversible investment, rather than on calendar time t as in a standard stochastic differential game.

It is worth pointing out that the use of Back and Paulsen’s zero-profit equilibrium as a punishment device to sustain a positive-profit equilibrium is analogous to employing the static Nash equilibrium of a prisoner’s dilemma to support cooperative outcomes in an infinitely repeated game (Fudenberg and Tirole, 1992). Recently, in their study of dynamic capital structure, Malenko and Tsoy (2025) adopt a similar approach: they use the Markov perfect equilibrium of DeMarzo and He (2021)—in which firms experience leverage ratcheting without benefits—to support a time-consistent equilibrium in which firms choose more prudent target leverage and avoid such ratcheting.

In a closely related paper, Steg (2024) studies the same irreversible investment game as we do. However, a firm’s admissible strategy in Steg (2024) requires information that is outside its natural filtration. Specifically, firm i ’s strategy in Steg (2024) is specified as a reflection-boundary rule that maps the current demand shock X_t and the rival’s capital stock K_t^{-i}

⁵To be precise, this is the continuous part of firm i ’s investment. Investment may also have a jump part. To ease exposition of the key mechanism, we focus on the continuous part.

into a prescribed capital level, above which firm i does nothing and below which it invests. This formulation requires firm i to condition its investment on its competitor's *simultaneous* investment. This is impossible because when choosing its own investment dK_t^i , firm i cannot observe dK_t^{-i} , as it is outside firm i 's information set at t . By contrast, in our model a firm's investment is adapted to its information filtration.

Our work is also related to the literature on repeated games with imperfect monitoring. A key contribution is Sannikov (2007), who formulates and solves the problem in continuous time by characterizing the set of continuation values achievable in public perfect equilibria via differential equations derived from the martingale representation theorem.

More broadly, our paper is related to another strand of the game-theoretic real-options literature that builds on stopping-time-based real-option models. Grenadier (1996) develops a duopoly real-options exercise model and applies it to real estate development. Dai et al. (2025) analyze a real-options game in which firms choose whether to enter as a leader or a follower, trading off first- versus second-mover advantages. In their setting, second-mover advantages give rise to mixed entry strategies. All these real-options game models feature optimal-stopping problems, whose single-agent formulations have been analyzed by McDonald and Siegel (1986) and Dixit and Pindyck (1994). In contrast, our duopolistic model features recurrent irreversible investments over time.

Models featuring recurrent irreversible investment have been widely studied. Abel and Eberly (1996) analyze a monopolist's optimal investment policies. In addition to the work by Grenadier (2002) and Back and Paulsen (2009) discussed earlier, Baldursson (1998) uses the same Nash, i.e., open-loop, equilibrium concept as Grenadier (2002) to study oligopoly games.⁶ Leahy (1993) and Baldursson and Karatzas (1996) study industry equilibrium in perfectly competitive settings. Kogan (2001, 2004) develop q -theoretic general-equilibrium asset-pricing models with irreversible investment.

There is also a growing body of literature that integrates industrial organization considerations into asset pricing models. Dou et al. (2022) extend the standard Lucas-tree asset pricing model to incorporate endogenous strategic competition. Chen et al. (2024) investigate

⁶Novy-Marx (2007) and Aguerrevere (2009) extend Grenadier's open-loop setting to analyze risk and return dynamics.

how strategic competition and financial distress interact dynamically.

2 Model

In this section, we introduce a classic real-option game-theoretic setting (Grenadier, 2002) in which firms compete by making recurrent irreversible investment decisions.

Consider a duopoly industry consisting of two firms: a and b . Let $\{K_t^i\}$ denote firm i 's capital stock, where $i = a, b$. Each firm uses its capital stock to produce output. Let $\pi^i(X_t, K_t^a, K_t^b)$ denote firm i 's profit at time t , where $\{X_t\}$ is an exogenous process representing the industry's demand shock. As in Grenadier (2002) and Back and Paulsen (2009), we assume that X_t follows a geometric Brownian motion (GBM):

$$dX_t = \mu X_t dt + \sigma X_t d\mathcal{B}_t, \quad X_0 > 0, \quad (1)$$

where $\{\mathcal{B}_t; t \geq 0\}$ is a one-dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with $\mathcal{B}_0 = 0$, μ is the drift and $\sigma > 0$ is the volatility parameter.

As in the literature, investment is irreversible in that $\{K_t^a\}$ and $\{K_t^b\}$ are nondecreasing over time. Let $\{M_t\}$ denote the running-maximum process for market demand $\{X_t\}$:

$$M_t = \max_{s \in [0, t]} X_s. \quad (2)$$

As investment irreversibility induces an option value of waiting, this running-maximum process plays a crucial role in our analysis, as we show later.

Let $Z_t = (X_t, K_t^a, K_t^b)$ denote the state, and let $\mathbb{Z} = (0, \infty) \times [k_0^a, \infty) \times [k_0^b, \infty)$ denote the state space, where $k_0^i \geq 0$ is firm i 's initial capital stock. For a given pair of capital stock processes $(\{K_t^a\}, \{K_t^b\})$, firm i 's value is

$$\mathbb{E}^{z_0} \left[\int_0^\infty e^{-rt} \left(\pi^i(Z_t) dt - p dK_t^i \right) \right], \quad (3)$$

where $\mathbb{E}^{z_0}[\cdot] = \mathbb{E}[\cdot \mid Z_0 = z_0]$ is the $t = 0$ expectation operator, $r > 0$ is a constant, and $p > 0$ is the constant marginal cost of purchasing a unit of capital. In the main body of our paper, we specify:⁷

$$\pi^i(x, k^a, k^b) = [x - (k^a + k^b)]k^i - ck^i, \quad (4)$$

⁷In Internet Appendix IA.C, we report the solution for the isoelastic demand case, another widely studied case in the literature, see, e.g., Grenadier (2002).

where $c > 0$ is a constant. We assume $\mu < r$ to ensure the finiteness of value functions.

3 Benchmarks: Monopoly and Open-loop Equilibrium

We report the monopoly solution in subsection 3.1 and the open-loop equilibrium in subsection 3.2. These results serve as useful benchmarks for our analysis of the closed-loop equilibrium.

3.1 Monopoly Solution (Abel and Eberly, 1996)

If the industry is instead a monopoly, then the monopolist makes profits at the rate of $\pi(x, k) = (x - k)k - ck$ and chooses its capital process $\{K_t; t \geq 0\}$ to maximize its value $\mathbb{E} \left[\int_0^\infty e^{-rt} \left([X_t - (K_t + c)]K_t dt - p dK_t \right) \right]$. Next, we summarize the monopolist's investment strategy (Abel and Eberly, 1996; Grenadier, 2002). Mathematically, the monopolist solves a singular control problem (Harrison, 1985).

Proposition 1 *The monopoly's optimal capital accumulation process, $\{K_t^{\text{mon}}\}$, is given by*

$$K_t^{\text{mon}} = \max \left\{ \frac{1}{2} \left(\frac{M_t}{\rho} - (rp + c) \right), k_0^{\text{mon}} \right\}, \text{ for all } t > 0, \quad (5)$$

where k_0^{mon} is the initial capital stock of the monopolist, M_t is the running maximum given in (2), $\rho > 0$ is a constant given by

$$\rho = \frac{\beta}{\beta - 1} \frac{r - \mu}{r}, \quad (6)$$

and $\beta > 2$ is a constant describing the optionality of the investment opportunity.⁸

$$\beta = \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}. \quad (7)$$

We can describe the monopoly's strategy as follows. First, if the demand is high in that $X_0 > \rho(2k_0^{\text{mon}} + rp + c)$, the monopolist instantly increases its capital stock at $t = 0$ from its initial level, k_0^{mon} , to $\frac{1}{2} \left[\frac{X_0}{\rho} - (rp + c) \right]$. Then for all $t \geq 0$, the monopolist does not invest until the running maximum M_t exceeds $\rho(2K_t^{\text{mon}} + rp + c)$, where K_t^{mon} is given in (5). Finally, after M_t reaches $\rho(2K_t^{\text{mon}} + rp + c)$, the monopolist invests if and only if M_t increases and the

⁸Here β is the positive root of the following (fundamental) quadratic equation: $\sigma^2 x(x - 1)/2 + \mu x - r = 0$. Convergence requires $\beta > 2$, which is equivalent to $r > 2\mu + \sigma^2$.

investment rate, dK_t^{mon} , is proportional to dM_t :

$$dK_t^{\text{mon}} = \frac{1}{2} \frac{dM_t}{\rho}. \quad (8)$$

For a given time interval dt , the running maximum $\{M_t\}$ increases, when both $X_t = M_t$ and $dX_t > 0$ hold. Under this scenario, $dM_t = dX_t > 0$. We can equivalently summarize the monopolist's capital accumulation process as follows: $K_t^{\text{mon}} = \inf\{k \geq k_0^{\text{mon}} \mid M_t \leq \mathcal{X}^{\text{mon}}(k)\}$ for $t > 0$, where $\mathcal{X}^{\text{mon}}(k)$ is the investment threshold function given by

$$\mathcal{X}^{\text{mon}}(k) = \rho(2k + rp + c). \quad (9)$$

To ease comparison with a closed-loop equilibrium solution, we report Back and Paulsen's open-loop equilibrium definition and the open-loop equilibrium solution.

3.2 Open-loop Equilibrium Definition and Solution

In an open-loop equilibrium, firms choose their strategies at $t = 0$. Specifically, at $t = 0$, firm i chooses its capital stock process $K^i \in \mathcal{S}_{\text{open}}^i(z_0)$ to maximize its market value given in (3), where $\mathcal{S}_{\text{open}}^i(z_0)$ is the set of firm i 's admissible open-loop strategies defined as:

$$\mathcal{S}_{\text{open}}^i(z_0) = \{\{K_t^i; t \geq 0\} \mid K_t^i \text{ is } \mathcal{F}_t\text{-adapted, nondecreasing, left-continuous, } K_0^i = k_0^i\}.$$

Note that firm i 's open-loop strategy only depends on the value of the initial state $Z_0 = z_0 := (x_0, k_0^a, k_0^b)$ and the demand-shock process $\{X_t; t \geq 0\}$, and importantly, does not depend on its competitor's strategy $\{K_t^{-i}; t > 0\}$. That is, firms compete by choosing their capital-stock paths at $t = 0$ and fully committing to their chosen paths. For a given pair, $K^a \in \mathcal{S}_{\text{open}}^a(z_0)$ and $K^b \in \mathcal{S}_{\text{open}}^b(z_0)$, let $\mathcal{J}^i(z_0; K^a, K^b)$ denote firm i 's value given in (3).

Back and Paulsen (2009) define an open-loop equilibrium as follows.

Definition 1 $K_{\text{open}}^a \in \mathcal{S}_{\text{open}}^a(z_0)$ and $K_{\text{open}}^b \in \mathcal{S}_{\text{open}}^b(z_0)$ constitute an *open-loop equilibrium* if the following conditions hold:

$$\mathcal{J}^a(z_0; K_{\text{open}}^a, K_{\text{open}}^b) \geq \mathcal{J}^a(z_0; K^a, K_{\text{open}}^b), \quad \forall K^a \in \mathcal{S}_{\text{open}}^a(z_0), \quad (10)$$

$$\mathcal{J}^b(z_0; K_{\text{open}}^a, K_{\text{open}}^b) \geq \mathcal{J}^b(z_0; K_{\text{open}}^a, K^b), \quad \forall K^b \in \mathcal{S}_{\text{open}}^b(z_0). \quad (11)$$

An open-loop equilibrium is a Nash equilibrium over entire time paths chosen at $t = 0$, and firms cannot revise their strategies in the future. Inequalities (10)-(11) hold only at $t = 0$.

Proposition 2 For $k_0^a = k_0^b$, there exists an open-loop equilibrium in which

$$K_{\text{open},t}^i = \max \left\{ \frac{1}{3} \left(\frac{M_t}{\rho} - (rp + c) \right), k_0^i \right\}, \text{ for all } t > 0, \quad (12)$$

where $\{M_t\}$ is the running-maximum process defined in (2) and $\rho > 0$ is given in (6).

Starting with the same level of the initial capital stock, firms adopt the same strategy so that their capital stocks are equal at all t . If the demand is high in that $X_0 > \rho(3k_0^i + rp + c)$, both firms instantly jump their capital stocks from k_0^i to $\frac{1}{3}[\frac{X_0}{\rho} - (rp + c)]$. Then for all $t \geq 0$, neither firm invests until the running maximum M_t exceeds $\rho(3K_{\text{open},t}^i + rp + c)$, where $K_{\text{open},t}^i$ is given in (12). After M_t reaches $\rho(3K_{\text{open},t}^i + rp + c)$, both firms invest whenever M_t increases. Finally, each firm invests at a rate proportional to dM_t :

$$dK_{\text{open},t}^i = \frac{1}{3} \frac{dM_t}{\rho}. \quad (13)$$

Therefore, the duopoly industry's investment rate equals $\frac{2}{3} \frac{dM_t}{\rho}$, which is faster than the monopoly's investment rate $\frac{1}{2} \frac{dM_t}{\rho}$ given in (8). This confirms the intuition that competition erodes the option value even in the open-loop equilibrium where firms compete by committing to their strategies at $t = 0$.

We can equivalently describe firm i 's capital process in the open-loop equilibrium using: $K_{\text{open},t}^i = \inf \{k \geq k_0^i : M_t \leq \mathcal{X}^{\text{open}}(k, k)\}$, where $\mathcal{X}^{\text{open}}(k^a, k^b)$ is the threshold given by

$$\mathcal{X}^{\text{open}}(k^a, k^b) := \begin{cases} \rho(2k^a + k^b + rp + c), & \text{if } k^a \leq k^b, \\ \rho(k^a + 2k^b + rp + c), & \text{if } k^a \geq k^b. \end{cases} \quad (14)$$

4 Closed-loop Equilibrium: Definition

In this section, we define closed-loop strategies and equilibria, and then provide a verification theorem. We proceed in three steps. First, in subsection 4.1, we define and characterize Back and Paulsen's closed-loop equilibrium in which firms make zero profits when investing. Then, in subsection 4.2, we define and characterize closed-loop equilibria in which firms make positive profits when investing, using Back and Paulsen's closed-loop equilibrium as a key building block. Finally, we provide a verification theorem for the closed-loop equilibria.

4.1 Back and Paulsen's Closed-loop Equilibrium with Zero Profits

Back and Paulsen (2009) identify a closed-loop equilibrium that features a perfectly competitive setting in which firms make zero profits when investing. In this equilibrium, which we refer to as the BP equilibrium, firms invest if and only if $X_t \geq \mathcal{X}^{\text{BP}}(K_t^a, K_t^b)$ where $\mathcal{X}^{\text{BP}}(k^a, k^b)$ is the investment threshold function given by:

$$\mathcal{X}^{\text{BP}}(k^a, k^b) = \rho(k^a + k^b + rp + c). \quad (15)$$

Back and Paulsen (2009) summarize this equilibrium using:

$$K_t^i = \inf \{k \geq K_0^i \mid \sup_{0 \leq s \leq t} [X_s - \mathcal{X}^{\text{BP}}(k, K_s^{-i})] \leq 0\}, \quad t > 0 \quad (16)$$

and point out that equation (16) “defines K^i as the smallest nondecreasing process such that $X_t \leq \mathcal{X}^{\text{BP}}(K_t^i, K_t^{-i})$ for all t .” Equation (16) implies that as long as (at least) one firm invests, the post-investment capital-stock pair, (K_t^a, K_t^b) , must lie on the investment threshold: $X_t = \mathcal{X}^{\text{BP}}(K_t^a, K_t^b) = \rho(K_t^a + K_t^b + rp + c)$ so that $K_t^a + K_t^b = \frac{X_t}{\rho} - (rp + c)$ holds.

We define the BP equilibrium using a sequential-game formulation in which the two firms are assigned *ex ante* the roles of leader and follower. In this formulation, the follower's closed-loop strategy at t is naturally a best response to the leader's strategy up to t . In Appendix A, we define this sequential game and characterize the leader's and the follower's closed-loop strategies. Below, we summarize the firms' capital-stock paths $(\{K_t^a\}, \{K_t^b\})$ and their value functions in the BP equilibrium. Without loss of generality, let firm a be the follower and firm b be the leader.

Theorem 1 *On the BP equilibrium path, the follower firm a 's capital stock remains constant: $K_t^a = k_0^a$ for all $t > 0$, and the leader firm b 's capital stock evolves as follows: $K_t^b = k_0^b + \left(\frac{M_t}{\rho} - (rp + c) - k_0^a - k_0^b\right)^+$ for all $t > 0$. Firm i 's value $V_{\text{BP}}^i(z)$ is given by*

$$V_{\text{BP}}^i(z) = k^i p, \quad x \geq \mathcal{X}^{\text{BP}}(k^a, k^b), \quad (17)$$

$$V_{\text{BP}}^i(z) = k^i \left[p + \frac{x}{r - \mu} - \frac{\mathcal{X}^{\text{BP}}(k^a, k^b)}{r\rho} \left(1 + \frac{1}{\beta - 1} \left(\frac{x}{\mathcal{X}^{\text{BP}}(k^a, k^b)} \right)^\beta \right) \right], \quad x < \mathcal{X}^{\text{BP}}(k^a, k^b), \quad (18)$$

where $z = (x, k^a, k^b)$, and $\mathcal{X}^{\text{BP}}(k^a, k^b)$ is the investment threshold given by (15).

In this BP equilibrium, if the demand is high in that $X_0 > \rho(k_0^a + k_0^b + rp + c)$, as the

leader, firm b instantly increases its capital stock from k_0^b to $\frac{X_0}{\rho} - (rp + c) - k_0^a$. Then for all $t \geq 0$, the leader does not invest until the running maximum M_t exceeds $\rho(K_t^b + k_0^a + rp + c)$ and whenever M_t reaches a new record, the leader increases its capital stock as follows:

$$dK_t^b = \frac{dM_t}{\rho}.$$

By contrast, as the follower, firm a never invests so that $K_t^a = k_0^a$ at all $t > 0$ on the equilibrium path. As we show in the next subsection, this BP equilibrium solution is a key building block for our definition of a closed-loop equilibrium in which firms make positive profits.

4.2 Closed-loop Equilibria with Positive Profits

We first define feasible closed-loop strategies and then define closed-loop equilibria.

4.2.1 Closed-loop Strategies

We decompose firm i 's investment over dt , dK_t^i , as follows:

$$dK_t^i = dK_t^{i,C} + \Delta K_t^i, \quad t \geq 0, \quad (19)$$

where $dK_t^{i,C}$ denotes the continuous component of investment dK_t^i and $\Delta K_t^i := K_{t+}^i - K_t^i$ denotes the instantaneous increase (jump) in K^i . We consider a set of closed-loop Markovian strategies. Assume that the jump component, ΔK_t^i , depends on the state, $Z_t = (X_t, K_t^a, K_t^b)$:

$$\Delta K_t^i = \mathbf{w}^i(Z_t), \quad (20)$$

where $\mathbf{w}^i(\cdot) \geq 0$ is a function of Z_t .

Inspired by the monopoly's strategy (8) and the open-loop equilibrium strategy (13), both of which are driven by the running-maximum process $\{M_t\}$, we conjecture and later verify that the continuous part of the closed-loop strategy, $dK_t^{i,C}$, can be written as:

$$dK_t^{i,C} = \mathbf{u}^i(Z_t) dM_t, \quad (21)$$

where $\mathbf{u}^i(\cdot) \geq 0$ is a function of Z_t . That is, firm i increases its capital stock continuously only if the running maximum of the demand shock, M_t , increases.

Definition 2 Let $\boldsymbol{\varphi}^i(Z_t) = (\mathbf{u}^i(Z_t), \mathbf{w}^i(Z_t))$. A pair of closed-loop strategies, $\boldsymbol{\varphi} = (\boldsymbol{\varphi}^a(\cdot), \boldsymbol{\varphi}^b(\cdot))$, is feasible if the following conditions hold:

(i) For any given $z = (x, k^a, k^b) \in \mathbb{Z}$, there exists a unique pair of $\mathcal{F}_t$ -adapted, nondecreasing, left-continuous processes, (K^a, K^b) , satisfying (19)-(21);

(ii) For any $z \in \mathbb{Z}$, $\mathbb{E}^z [|K_T^i|^n] < \infty$ for any finite $T > 0$ and $n > 1$, and

$$\mathbb{E}^z \left[\int_0^\infty e^{-rt} |\pi^i(X_t, K_t^a, K_t^b)| dt \right] < \infty. \quad (22)$$

Let $\mathcal{S}$ denote the set of all feasible (closed-loop) strategies.

For any $z \in \mathbb{Z}$ and any feasible closed-loop strategy pair $\varphi \in \mathcal{S}$, we define firm i 's value as:

$$J^i(z; \varphi^a, \varphi^b) = \mathbb{E}_t^z \left[\int_t^\infty e^{-r(s-t)} \left(\pi^i(X_s, K_s^{a,\varphi}, K_s^{b,\varphi}) ds - p dK_s^{i,\varphi} \right) \right], \quad (23)$$

where $\mathbb{E}_t^z[\cdot] = \mathbb{E}_t[\cdot | Z_t = z]$ and $(K^{a,\varphi}, K^{b,\varphi})$ is the associated capital-stock process pair.

Now we are ready to define closed-loop equilibria for our capital accumulation game that involves singular controls.

4.2.2 Closed-loop Equilibria

Our equilibrium definition imposes two conditions. The first condition makes the equilibrium impervious to preemption: the additional value created by expanding capital cannot exceed its purchase cost. The second condition uses reversion to the BP zero-profit closed-loop equilibrium as a punishment to support a closed-loop equilibrium with positive profits.

Definition 3 A strategy pair $\varphi_* = (\varphi_*^a, \varphi_*^b) \in \mathcal{S}$ constitutes a *closed-loop equilibrium* if for any $Z_t = z = (x, k^a, k^b) \in \mathbb{Z}$, the following conditions hold for $J^a(z; \varphi_*)$:

(i) for any $\lambda^a > 0$, the following inequality holds:

$$J^a(x, k^a, k^b; \varphi_*, \varphi_*^b) \geq J^a(x, k^a + \lambda^a, k^b; \varphi_*, \varphi_*^b) - p\lambda^a; \quad (24)$$

(ii) for any $\varphi = (\varphi^a, \varphi_*^b) \in \mathcal{S}$,

$$J^a(z; \varphi_*, \varphi_*^b) \geq \mathbb{E}_t^z \left[\int_t^{t+\varepsilon} e^{-r(s-t)} \left(\pi^a(Z_s^\varphi) ds - p dK_s^{a,\varphi} \right) + e^{-r\varepsilon} V_{BP}^a(Z_{t+\varepsilon}^\varphi) \right], \quad (25)$$

for all sufficiently small $\varepsilon > 0$, where $Z_s^\varphi = (X_s, K_s^{a,\varphi}, K_s^{b,\varphi})$.

Symmetric conditions are required to hold for $J^b(z; \varphi_*, \varphi_*^b)$. Let $V^a(z) = J^a(z; \varphi_*, \varphi_*^b)$ and $V^b(z) = J^b(z; \varphi_*^a, \varphi_*)$ denote the closed-loop equilibrium value functions for a and b , respectively.

Inequality (24) is a no-preemption condition. To see this, note that the right side of (24) represents firm a 's net payoff if it instantaneously increases its capital stock by $\lambda^a > 0$ and its rival does not respond to the deviation but simply continues to play φ_*^b from the new state. Condition (i) thus requires that a firm cannot gain from a lumpy investment even if the deviation goes unpunished. It is the analogue in our setting of the requirement, conjectured by Back and Paulsen, that equilibrium strategies be “impervious to preemption.” As Theorem 2 below shows, condition (i) is equivalent to the requirement that firm a 's marginal q never exceed the price of capital p , which is what gives our equilibria their q -theoretic structure.

Inequality (25) describes the equilibrium condition that there is no profitable one-shot deviation by firm a . To see this, fix a small $\varepsilon > 0$ and suppose that firm a deviates from its equilibrium strategy φ_*^a to another strategy φ^a over the short period $[t, t + \varepsilon)$. Then, firm b penalizes firm a by switching from its equilibrium strategy φ_*^b to the leader's strategy in the BP equilibrium given in Theorem 1. Firm b 's switching of its strategy is feasible as it observes whether firm a has deviated from its equilibrium strategy φ_*^a or not in the $[t, t + \varepsilon)$ period, and it is credible because the BP continuation is itself an equilibrium (Theorem 5 in Appendix A). Immediately after firm b 's punishment, firm a plays the follower's strategy in the BP equilibrium (again see Theorem 1). For other applications of the one-shot deviation principle in continuous time, see, for example, Sannikov (2007) for repeated games under imperfect monitoring and Malenko and Tsoy (2025) for dynamic capital structure analysis.

We emphasize that the threat of reversion to the BP closed-loop equilibrium plays no role in deterring lumpy investment. The payoff comparison in condition (i) allows for the possibility that firm b does not observe firm a 's deviation when it occurs and, even if the deviation becomes apparent afterward, does not punish firm a . Even anticipating this absence of punishment, firm a cannot gain from an additional lumpy investment, as required by condition (i). Deterring a firm from using lumpy investment to preempt its rival thus does not rely on the threat of reversion to the BP closed-loop equilibrium: the condition that marginal q not exceed p suffices.⁹

Next, we state a verification theorem for a closed-loop equilibrium in which firms earn positive profits.

⁹We thank Kerry Back for the question that led us to write this paragraph.

Theorem 2 *A strategy pair $(\varphi_*^a, \varphi_*^b) \in \mathcal{S}$ constitutes a closed-loop equilibrium in which firms earn positive profits if firm value $V^i(z) = J^i(z; \varphi_*^a, \varphi_*^b)$ for $i = a, b$ is differentiable and satisfies*

$$\frac{\partial V^i(z)}{\partial k^i} \leq p, \quad \text{for any } z \in \mathbb{Z} \quad (26)$$

and

$$V^i(z) > V_{\text{BP}}^i(z), \quad \text{for any } z \in \mathbb{Z}. \quad (27)$$

Inequality (26) is equivalent to condition (24), stating that, in a closed-loop equilibrium, firm i 's marginal q cannot exceed the marginal cost of acquiring capital, p . In addition, inequality (27) implies condition (25), so that the strategy pair is a closed-loop equilibrium in which firms earn positive profits, in the sense that firm value strictly exceeds its value $V_{\text{BP}}^i(z)$ under perfect competition, which equals the replacement cost of capital, $k^i p$, whenever $x \geq \mathcal{X}^{\text{BP}}(k^a, k^b)$.

Next, we characterize the closed-loop equilibrium solution in Sections 5 and 6.

5 Closed-loop Equilibrium Solution Characterization

In this section, we first conjecture that a closed-loop equilibrium solution features four regions (subsection 5.1) and then analyze general properties for the solution (subsection 5.2).

5.1 Four-region Solution

At each point (x, k^a, k^b) in the state space, a firm either invests or not. We therefore conjecture that the solution can be characterized by four mutually exclusive regions. First, consider the region where neither firm invests. Let $\mathbb{N}_{ab}$ denote this region:

$$\mathbb{N}_{ab} = \{(x, k^a, k^b) \in \mathbb{R}_+^3 : x < \mathcal{X}(k^a, k^b)\}, \quad (28)$$

where $\mathcal{X}(k^a, k^b)$ is an endogenously determined equilibrium investment boundary.¹⁰ Suppose demand X_t is low relative to the industry capital stock in that $X_t < \mathcal{X}(K_t^a, K_t^b)$. Then, it is optimal for firms to wait and invest only when $\{X_s\}_{s>t}$ rises in the future, exceeding the

¹⁰We only consider the investment boundary $\mathcal{X}(k^a, k^b)$ that is continuous and increasing in both k^a and k^b , and symmetric, i.e., $\mathcal{X}(k^a, k^b) = \mathcal{X}(k^b, k^a)$.

threshold $\mathcal{X}(K_t^a, K_t^b)$. Now consider the region where firm i invests. Let $\mathbb{I}_i$ denote this region:

$$\mathbb{I}_i = \{(x, k^a, k^b) \in \mathbb{R}_+^3 : x \geq \mathcal{X}(k^a, k^b), x \geq \mathcal{X}(k^i, k^i)\}. \quad (29)$$

Note that $\mathbb{I}_i$ requires two conditions. The first condition $X_t \geq \mathcal{X}(K_t^a, K_t^b)$ implies that at least one firm invests, as market demand X_t is larger than the investment threshold $\mathcal{X}(K_t^a, K_t^b)$. The second condition $X_t \geq \mathcal{X}(K_t^i, K_t^i)$ implies that firm i must invest at t .¹¹

In summary, we conjecture and later verify that the solution features four mutually exclusive regions: (1) $\mathbb{N}_{ab}$ where neither firm invests, (2) $\mathbb{I}_a \setminus \mathbb{I}_b$ where only firm a invests, (3) $\mathbb{I}_b \setminus \mathbb{I}_a$ where only firm b invests, and (4) $\mathbb{I}_a \cap \mathbb{I}_b$, the intersection of $\mathbb{I}_a$ and $\mathbb{I}_b$, where both firms invest.¹² Projecting the three-dimensional equilibrium-solution regions described above onto the $(K_t^a, K_t^b) = (k^a, k^b)$ plane for a given demand shock $X_t = x > 0$, we obtain the four-region solution described in Figure 1. Strictly above the investment boundary (the ACE curve) is the inaction region, which is the projection of $\mathbb{N}_{ab}$, where neither firm invests:

$$\mathbb{N}_{ab}(x) = \{(k^a, k^b) \in \mathbb{R}_+^2 : x < \mathcal{X}(k^a, k^b)\}. \quad (30)$$

Elsewhere in the first quadrant, we have the other three regions: a.) the ABC region where only firm a invests; b.) the CDE region where only firm b invests; and c.) the OBCD (square) region where both firms invest.

Note that when market demand is high so that the industry expands, the smaller firm always invests. The intuition is that the smaller firm benefits more from investing due to the diminishing marginal return.¹³ It is this ability to condition investment on the rival's capital stock that distinguishes closed-loop from open-loop strategies.

5.2 Characterizing General Properties of the Solution

Below, we first characterize firm i 's marginal q and its competitor's cross marginal q in the region when firm i invests in subsection 5.2.1, and then solve the partial differential equation in closed form for firm i 's value subject to a given boundary in subsection 5.2.2.

¹¹This result follows from two equilibrium properties: (1) it is always the smaller firm that invests in equilibrium and (2) the investment boundary $\mathcal{X}(K_t^a, K_t^b)$ increases in a firm's capital stock.

¹²As we show below in equilibrium firm i undertakes lumpy investment only at time 0, when (x_0, k_0^a, k_0^b) lies in the interior of $\mathbb{I}_i$. At $t > 0$, firms either engage in continuous investment or take no action, so that the evolution of (X_t, K_t^a, K_t^b) always lies within the closure of $\mathbb{N}_{ab}$ for all $t > 0$.

¹³It is only in the square region where both firms invest in equilibrium. This is because the marginal product is high enough even for the larger firm in this region.

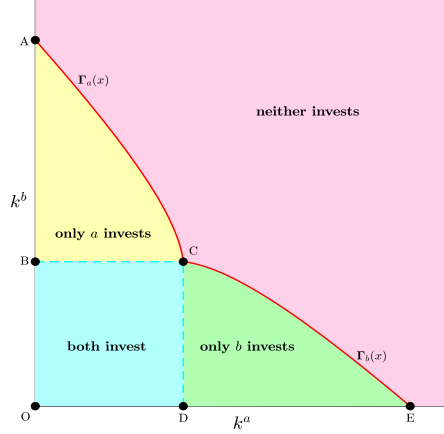


Figure 1: EQUILIBRIUM SOLUTION REGIONS ASSOCIATED WITH AN INVESTMENT BOUNDARY ACE. Inaction region $\mathbf{N}_{ab}(x)$ is given in (30) and firm i 's investment boundary is $\Gamma_i(x) := \{(k^a, k^b) \in \mathbb{R}_+^2 : \mathcal{X}(k^a, k^b) = x, k^i \leq k^{-i}\}$.

5.2.1 Firm i 's Marginal q and Firm $-i$'s Cross Marginal q when Firm i Invests

When firm i invests, its marginal q must equal the marginal cost of purchasing capital p . This is the standard first-order condition as in a classic q theory of investment for a single firm (Abel and Eberly, 1996). Also note that its competitor's value function cannot change when firm i invests. That is, the sensitivity of $V^{-i}(z)$ with respect to k^i , which we refer to as firm $-i$'s *cross marginal* q , must be zero when firm i invests. This is because firm i can instantly increase its capital stock when investing at t and firm $-i$'s value must satisfy the value-matching condition. In summary, we have:

$$\frac{\partial V^a(z)}{\partial k^a} = p \quad \text{and} \quad \frac{\partial V^b(z)}{\partial k^a} = 0 \quad \text{in ABC where firm } a \text{ invests,} \quad (31)$$

$$\frac{\partial V^b(z)}{\partial k^b} = p \quad \text{and} \quad \frac{\partial V^a(z)}{\partial k^b} = 0 \quad \text{in CDE where firm } b \text{ invests,} \quad (32)$$

$$\frac{\partial V^a(z)}{\partial k^a} = \frac{\partial V^b(z)}{\partial k^b} = p \quad \text{and} \quad \frac{\partial V^b(z)}{\partial k^a} = \frac{\partial V^a(z)}{\partial k^b} = 0 \quad \text{in OBCD where both invest.} \quad (33)$$

5.2.2 Characterizing Firm i 's Value, $V^i(z)$, for a Given Investment Boundary

Having characterized the general properties in the regions where at least one firm invests, we now turn to the region $\mathbf{N}_{ab}(x)$ where neither firm invests: $x < \mathcal{X}(k^a, k^b)$. Firm i 's value,

$V^i(z)$, satisfies the following HJB equation as in standard real-option settings:

$$rV^i(z) = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V^i(z)}{\partial x^2} + \mu x \frac{\partial V^i(z)}{\partial x} + \pi^i(z), \quad (34)$$

where $\pi^i(z)$ is the profit function given in (4). The general solution for (34) takes the form:

$$V^i(z) = \Psi^i(z) + H^i(k^a, k^b)x^\beta, \quad (35)$$

where $\Psi^i(z)$ is the value of firm i 's assets in place given by

$$\Psi^i(z) = \mathbb{E}_t^z \left[\int_t^\infty e^{-r(s-t)} \pi^i(X_s, k^a, k^b) ds \right] = \left(\frac{x}{r-\mu} - \frac{k^a + k^b + c}{r} \right) k^i \quad (36)$$

and $H^i(k^a, k^b)$ measures firm i 's option value under imperfect competition, which we determine below. Note that $\Psi^i(z)$ is the special solution for (34) and would be firm i 's value under the counterfactual that the industry has no growth from t onward: $dK_s^a = dK_s^b = 0$ for all $s \geq t$. That is, $\Psi^i(z)$ measures the value of firm i 's assets in place. Next, we solve for $H^a(k^a, k^b)$.

Equations (31) and (32) imply the following boundary conditions:

$$\frac{\partial V^a(z)}{\partial k^a} = p \quad \text{when } \mathcal{X}(k^a, k^b) = x \text{ and } k^a \leq k^b, \quad (37)$$

$$\frac{\partial V^a(z)}{\partial k^b} = 0 \quad \text{when } \mathcal{X}(k^a, k^b) = x \text{ and } k^a \geq k^b. \quad (38)$$

Substituting (35) into (37) and (38), and then simplifying the algebra, we obtain the following result. First, if firm a is the small one: $k^a \leq k^b$, we have

$$\frac{\partial H^a(k^a, k^b)}{\partial k^a} = \left(\frac{2k^a + k^b + rp + c}{r} - \frac{\mathcal{X}(k^a, k^b)}{r-\mu} \right) \mathcal{X}(k^a, k^b)^{-\beta}. \quad (39)$$

Second, if firm a is the larger one: $k^a \geq k^b$, we have

$$\frac{\partial H^a(k^a, k^b)}{\partial k^b} = \frac{k^a}{r} \mathcal{X}(k^a, k^b)^{-\beta}. \quad (40)$$

In addition, we have the boundary condition: $\lim_{k \rightarrow \infty} H^a(k, k) = 0$. This is because as the industry becomes very large, neither firm will ever invest and the option value of investing is zero. Using (39), (40), and $\lim_{k \rightarrow \infty} H^a(k, k) = 0$, and integrating along the 45° line in the (k^a, k^b) plane, we obtain the following closed-form solution on the 45° line:

$$H^a(k, k) = - \int_k^\infty \left(\frac{4\alpha + rp + c}{r} - \frac{\mathcal{X}(\alpha, \alpha)}{r-\mu} \right) \mathcal{X}(\alpha, \alpha)^{-\beta} d\alpha. \quad (41)$$

Finally, using (41) for $H^a(k, k)$ to integrate (39) and (40), we obtain the following expressions for $H^a(k^a, k^b)$, firm a 's option value off the 45° line:

$$H^a(k^a, k^b) = H^a(k^b, k^b) - \int_{k^a}^{k^b} \left(\frac{2\alpha + k^b + rp + c}{r} - \frac{\mathcal{X}(\alpha, k^b)}{r-\mu} \right) \mathcal{X}(\alpha, k^b)^{-\beta} d\alpha, \quad k^a \leq k^b,$$

$$H^a(k^a, k^b) = H^a(k^a, k^a) - \int_{k^b}^{k^a} \frac{k^a}{r} \mathcal{X}(k^a, \alpha)^{-\beta} d\alpha, \quad k^a \geq k^b. \quad (42)$$

Summary. We have thus far characterized the following properties of a closed-loop equilibrium, given an investment boundary $\mathcal{X}(k^a, k^b)$. Firm i 's value, $V^i(z)$, satisfies (35) subject to (37)-(38) in the inaction region where neither firm invests, (31)-(33) in the other three regions where at least one firm invests, and the inequalities (26)-(27) in all four regions. To fully characterize the closed-loop equilibrium, we still have to determine the equilibrium investment boundary $\mathcal{X}(k^a, k^b)$, to which we turn next.

6 Closed-loop Equilibria with Piecewise Linear Triggers

To complete our analysis in the preceding section and obtain a set of closed-loop equilibria, we first conjecture and verify that the investment boundary is piecewise linear in subsection 6.1, and then derive closed-form solutions for these closed-loop equilibria in subsection 6.2.

6.1 Closed-loop Equilibria with Piecewise Linear Thresholds

Inspired by the piecewise linear boundary given in (14) for the open-loop equilibrium (Grenadier, 2002; Back and Paulsen, 2009), we conjecture and then verify that there exists a set of closed-loop equilibria, also characterized by piecewise linear investment boundaries, but with slopes and intercepts different from those in the open-loop equilibrium. Specifically, we conjecture and verify that the boundary $\mathcal{X}(k^a, k^b)$ takes the following symmetric form:

$$\mathcal{X}(k^a, k^b) := \begin{cases} \rho(\theta_2 k^a + \theta_1 k^b + rp + c), & \text{if } k^a \leq k^b, \\ \rho(\theta_1 k^a + \theta_2 k^b + rp + c), & \text{if } k^a \geq k^b, \end{cases} \quad (43)$$

where (θ_1, θ_2) is a pair of positive real numbers from a set Θ to be determined. Note the symmetry between the $k^a \leq k^b$ and $k^b \leq k^a$ subspaces: θ_2 is the coefficient multiplying the capital stock of the smaller firm and θ_1 is the coefficient multiplying the capital stock of the larger firm. We solve the associated firm values, $V^a(z)$ and $V^b(z)$, in closed form by substituting the conjectured boundary (43) into (41)-(42).

Using (41)-(43) to simplify (26) and (27) for firm a (and by symmetry conducting the same exercise for firm b), we obtain the following conditions for (θ_1, θ_2) appearing in $\mathcal{X}(k^a, k^b)$.

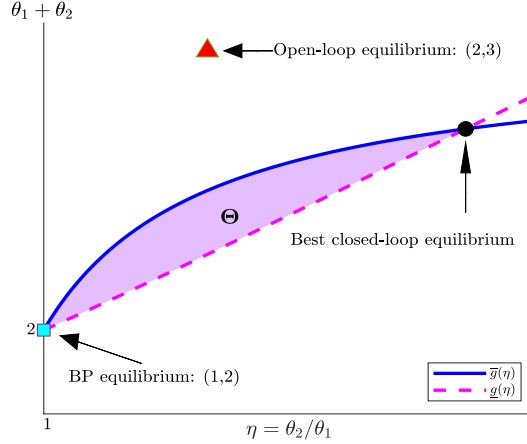


Figure 2: CHARACTERIZING THE SET OF ALL CLOSED-LOOP EQUILIBRIA WITH PIECEWISE LINEAR TRIGGERS USING THE ADMISSIBLE SET Θ FOR (θ_1, θ_2) . The area, bounded by the solid blue line $\bar{g}(\cdot)$ given in (45) and the dashed magenta line $\underline{g}(\cdot)$ given in (46), defines a set Θ for (θ_1, θ_2) with $\theta_2 \geq 1 \geq \theta_1 > 0$. A pair $(\theta_1, \theta_2) \in \Theta$ (including those on the boundary) corresponds to a closed-loop equilibrium.

The BP equilibrium is the one with the lowest firm value, corresponding to the cyan square: $\theta_1 = \theta_2 = 1$. The closed-loop equilibrium with the highest firm value, referred to as the best equilibrium, corresponds to the black dot. The open-loop equilibrium corresponds to the red triangle, which is outside of the set Θ .

Condition 1 *The pair (θ_1, θ_2) , with $\theta_2 \geq \theta_1 > 0$, satisfies the following inequality bounds:*

$$\underline{g}(\theta_2/\theta_1) \leq \theta_1 + \theta_2 \leq \bar{g}(\theta_2/\theta_1), \quad (44)$$

where the upper and lower bound functions, $\bar{g}(\cdot)$ and $\underline{g}(\cdot)$, are respectively given by:

$$\bar{g}(\eta) = 3 - 1/\eta \quad \text{and} \quad (45)$$

$$\underline{g}(\eta) = g \left(1 + \frac{(1 + \eta)^\beta - 1}{\eta} \right). \quad (46)$$

Here, $g(\cdot)$ is the inverse function of $\check{g}(s) = s^\beta + \frac{\beta}{\beta-2}(s-2)$ for $s \geq 2$.

Let Θ denote the set of pairs (θ_1, θ_2) satisfying **Condition 1**. The right inequality in (44) is the no-preemption (marginal q) condition (26), and the left inequality is the punishment condition (27): Corollary IA.1 in Internet Appendix IA.E shows that each inequality is necessary for the corresponding condition, and Lemmas 2 and 4 show that together they are sufficient.

In Figure 2, the shaded area, bounded by the solid blue line $\bar{g}(\cdot)$ given in (45) and the dashed magenta line $\underline{g}(\cdot)$ given in (46), defines a closed set Θ for (θ_1, θ_2) with $\theta_2 \geq \theta_1 > 0$. Next, we characterize the properties of the set of closed-loop equilibria with piecewise-linear

triggers.

Lemma 1 *The equation $\bar{g}(\eta) - \underline{g}(\eta) = 0$ has a unique root η^* in the domain $\eta > 1$. Moreover, $\bar{g}(\eta) > \underline{g}(\eta)$ for any $\eta \in (1, \eta^*)$, and $\bar{g}(\eta) < \underline{g}(\eta)$ for any $\eta > \eta^*$.*

*Under **Condition 1**, $\theta_1 + \theta_2$ attains its maximum if and only if $(\theta_1, \theta_2) = (\theta_1^*, \theta_2^*)$, and attains its minimum if and only if $(\theta_1, \theta_2) = (1, 1)$, where*

$$\theta_1^* = \frac{3 - 1/\eta^*}{1 + \eta^*} \quad \text{and} \quad \theta_2^* = \frac{3\eta^* - 1}{1 + \eta^*}. \quad (47)$$

Furthermore, for any $(\theta_1, \theta_2) \in \Theta \setminus \{(1, 1)\}$, we have $\theta_2 > 1 > \theta_1 > 0$ and $2 < \theta_1 + \theta_2 < 3$.

The equilibrium in the set Θ with the highest firm value, which we also refer to as the best closed-loop equilibrium (see the black dot in Figure 2), is uniquely characterized by (θ_1^*, θ_2^*) given in (47).

The BP equilibrium is the one with the lowest firm value, corresponding to the cyan square in Figure 2: $\theta_1 = \theta_2 = 1$. Note that the open-loop equilibrium corresponds to the red triangle, which is outside of the set Θ . This is because the open-loop trigger violates the no-preemption condition (i) in Definition 3: there exists a set of (K_t^a, K_t^b) at time t where a firm's marginal q exceeds the purchase cost of capital p , so that a firm could gain by a lumpy investment. (The open-loop equilibrium is not subgame perfect for the reason discussed in the Introduction; the point here is that its investment path cannot be supported by closed-loop strategies in the class defined by conditions (i)–(ii).)

In Figure 3, fixing time t , $X_t = x$, and $(K_t^a, K_t^b) = (k^a, k^b)$, we illustrate how the investment threshold in a closed-loop equilibrium differs from that in the open-loop equilibrium on the (k^a, k^b) plane. By symmetry, we focus on the upper half of the first quadrant, where firm a is smaller than firm b , i.e., $k^a \leq k^b$.

First, because competition is fiercer in a closed-loop equilibrium (due to subgame perfection) than in the open-loop equilibrium, the investment threshold in the former lies to the right of that in the latter.

Second, in a closed-loop equilibrium associated with (θ_1, θ_2) , the investment threshold given in (43) is piecewise linear, with slope $-\frac{\theta_2}{\theta_1}$, whose absolute value exceeds one (the AB line segment). Using the expression for the investment threshold in a closed-loop equilibrium, $x = \mathcal{X}(k^a, k^b)$, where $\mathcal{X}(\cdot, \cdot)$ is given in (43), we obtain the coordinates of point A on the

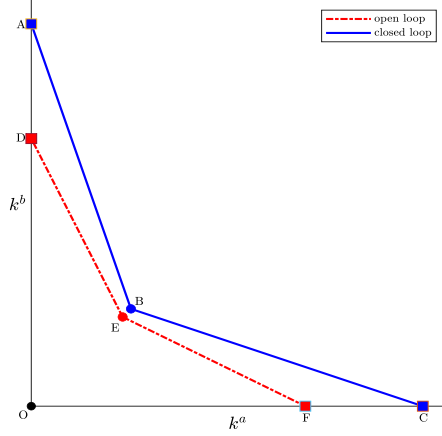


Figure 3: PIECEWISE LINEAR INVESTMENT THRESHOLDS IN DIFFERENT EQUILIBRIA. Fixing time t , $X_t = x$ and $(K_t^a, K_t^b) = (k^a, k^b)$, this figure plots investment thresholds at t in the open-loop equilibrium (DEF), and a (representative) closed-loop equilibrium (ABC) on the (k^a, k^b) plane.

vertical axis: $(k^a, k^b) = \left(0, \frac{x/\rho - (rp+c)}{\theta_1}\right)$, and the coordinates of point B on the 45° line: $k^a = k^b = \frac{x/\rho - (rp+c)}{\theta_1 + \theta_2}$.

Third, in the open-loop equilibrium, the investment threshold (the DE segment) has slope -2 , as $\theta_1 = 1$ and $\theta_2 = 2$. Using the expression for the open-loop investment threshold, $x = \mathcal{X}^{\text{open}}(k^a, k^b)$, yields the coordinates of point E on the 45° line: $k^a = k^b = \frac{x/\rho - (rp+c)}{3}$. Because $\theta_1 + \theta_2 < 3$ (see Lemma 1), point E lies to the southwest of point B. This property again reflects the prediction that firms compete more fiercely under the closed-loop equilibrium than under the open-loop equilibrium, as a lower value of $\theta_1 + \theta_2$ implies more intense competition.

6.2 Closed-loop Equilibrium Solution

Next, we summarize the closed-loop equilibrium strategy (see Appendix B for a proof.)

Theorem 3 *Under Condition 1, for the investment boundary $\mathcal{X}(k^a, k^b)$ given in (43), a strategy pair $(\varphi_*^a, \varphi_*^b) = \{(\mathbf{u}_*, \mathbf{w}_*), (\mathbf{u}_*, \mathbf{w}_*)\} \in \mathcal{S}$ constitutes a closed-loop equilibrium if*

$$\mathbf{u}_*^i(z) = \left[\frac{\mathbf{1}_{k^i < k^{-i}}}{\rho\theta_2} + \frac{\mathbf{1}_{k^i = k^{-i}}}{\rho(\theta_1 + \theta_2)} \right] \mathbf{1}_{x \geq \mathcal{X}(k^a, k^b)}, \quad (48)$$

$$\mathbf{w}_*^i(z) = \inf \{ \delta \geq 0 : \mathcal{X}(k^i + \delta, \max\{k^{-i}, \mathcal{K}(x)\}) \geq x \}, \quad (49)$$

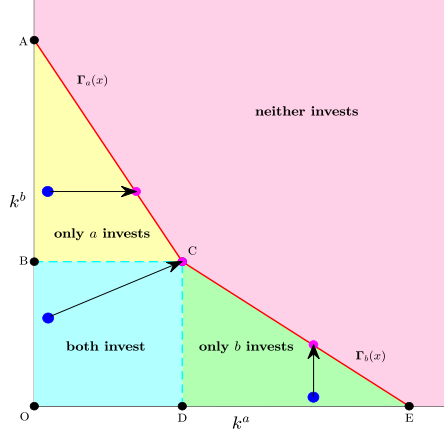


Figure 4: EQUILIBRIUM SOLUTION REGIONS WHEN THE INVESTMENT BOUNDARY IS PIECEWISE LINEAR.

for all $z = (x, k^a, k^b) \in \mathbb{Z}$, where $\mathbf{1}_{\{\cdot\}}$ is an indicator function and $\mathcal{K}(x)$ is given by

$$\mathcal{K}(x) = \frac{1}{\theta_1 + \theta_2} \left[\frac{x}{\rho} - (rp + c) \right]. \quad (50)$$

Next, we use Figure 4 to illustrate the industry equilibrium dynamics in a given closed-loop equilibrium. Depending on where the state vector Z_t is, there are four scenarios starting at t .

First, if the economy is located in the region northeast of the ACE boundary (excluding the boundary itself), neither firm invests. Second, if the initial state lies within the OBCD square region (including its boundaries), both firms invest so that point C in Figure 4 is instantly reached: $K_t^a = K_t^b = \mathcal{K}(M_t)$. Thereafter, the firms invest along the 45° line, with the same increments proportional to dM_t :

$$dK_t^a = dK_t^b = \frac{dM_t}{\rho(\theta_1 + \theta_2)}. \quad (51)$$

The sum $\theta_1 + \theta_2$ dictates the rate at which firms invest. Equation (51) shows that the larger the sum the slower firms invest, and the more option value they preserve (see Section 7 for details). The pair (θ_1^*, θ_2^*) , which attains the highest value of $\theta_1 + \theta_2$ in the set Θ , is associated with the closed-loop equilibrium with the highest firm value, referred to as the best closed-loop equilibrium (see the black dot in Figure 2).

Third, if the initial state lies within the ABC triangle region, firm a increases its capital to the AC boundary in Figure 4, while firm b remains inactive. Afterwards, firm a 's investment

is proportional to dM_t , whereas firm b does not invest:

$$dK_t^a = \frac{dM_t}{\rho\theta_2} \quad \text{and} \quad dK_t^b = 0. \quad (52)$$

This process continues until point C in Figure 4 is reached. Once point C is reached, both firms invest along the 45° line as described in the above scenario. Finally, by symmetry, the same analysis applies if the initial state lies within the CDE triangle region. In this case, we simply switch the strategies of firms a and b .

These four scenarios are endogenously connected over time through the stochastically evolving ACE boundary: $X_t = \mathcal{X}(K_t^a, K_t^b)$.

Below, we report the associated equilibrium firm values in a given closed-loop equilibrium in Proposition 3 (see Appendix B for a proof.)

Proposition 3 *In an equilibrium characterized in Theorem 3, $V^i(z)$ for $i = a, b$ is given as follows:*

1. *in the region where $x \leq \mathcal{X}(k^a, k^b)$, $V^i(z)$ is given by (35);*
2. *in the region where $k^a \leq \mathcal{K}(x)$ and $k^b \leq \mathcal{K}(x)$, $V^i(z) = V^i(x, \mathcal{K}(x), \mathcal{K}(x)) - p(\mathcal{K}(x) - k^i)$;*
3. *in the region where $x \geq \mathcal{X}(k^a, k^b)$ and $k^b > \mathcal{K}(x)$, $V^a(z) = V^a(x, \hat{k}^a, k^b) - p(\hat{k}^a - k^a)$ and $V^b(z) = V^b(x, \hat{k}^a, k^b)$, where $V^i(x, \hat{k}^a, k^b)$ is given in (35) and $\hat{k}^a$ satisfies*

$$\mathcal{X}(\hat{k}^a, k^b) = x. \quad (53)$$

By symmetry, we can calculate $V^i(z)$ in the region where $x \geq \mathcal{X}(k^a, k^b)$ and $k^a > \mathcal{K}(x)$ by switching the roles of firms a and b .

Next, we establish a key result for cross marginal q .

Proposition 4 *In the closed-loop equilibrium characterized in Theorem 3, a firm's cross marginal q is weakly negative. Specifically, for each firm $i = a, b$, the marginal effect of the rival firm's capital stock on firm i 's value is nonpositive for all $z \in \mathbb{Z}$; that is,*

$$\frac{\partial V^i(z)}{\partial k^{-i}} \leq 0. \quad (54)$$

See Appendix B for a proof. This result is intuitive. A higher level of rival capital, K^{-i} , raises total industry capital and output, thereby lowering the output price and making firm i weakly worse off.

7 A Duopolistic q Theory of Investment

In this section, we develop a duopolistic q theory of investment by using the best closed-loop equilibrium solution for the real-option game analyzed earlier. We focus on a symmetric setting where the two firms have the same initial capital stock: $k_0^a = k_0^b$.

7.1 Industry Investment Dynamics and Valuation

Let $V_{\text{closed}}^{\text{duo}}(z) = V^a(z) + V^b(z)$ denote the value of a duopoly industry in a closed-loop equilibrium where $z := (x, k, k)$. The industry value in the Back–Paulsen solution, $V_{\text{BP}}^{\text{duo}}(z)$, is associated with the strategy pair $(\theta_1, \theta_2) = (1, 1)$. The industry value in the best closed-loop equilibrium, $V_{\text{best}}^{\text{duo}}(z)$, is associated with the strategy pair (θ_1^*, θ_2^*) given in (47) and is the highest among all closed-loop equilibria. As a reference, we also report the industry value in the open-loop equilibrium, denoted by $V_{\text{open}}^{\text{duo}}(z)$. Below we summarize the rank order of these industry values and industry investment rates.

Proposition 5 *For any $(\theta_1, \theta_2) \in \Theta$, the following inequalities hold:*

$$V^{\text{mon}}(x, 2k) > V_{\text{open}}^{\text{duo}}(x, k, k) > V_{\text{best}}^{\text{duo}}(x, k, k) \geq V_{\text{closed}}^{\text{duo}}(x, k, k) \geq V_{\text{BP}}^{\text{duo}}(x, k, k), \quad (55)$$

where V^{mon} denotes the monopoly value. Finally, the industry value in a closed-loop equilibrium, $V_{\text{closed}}^{\text{duo}}(x, k, k)$, is a strictly increasing function of $\theta_1 + \theta_2$.

Consistent with our intuition, as competition is fiercest in the BP closed-loop equilibrium where $(\theta_1, \theta_2) = (1, 1)$, the industry value $V_{\text{BP}}^{\text{duo}}(x, k, k)$ is the lowest. The industry value in the open-loop equilibrium is larger than in all closed-loop equilibria including the best closed-loop equilibrium associated with (θ_1^*, θ_2^*) . This is because, by definition, unlike in a closed-loop equilibrium, a firm in the open-loop equilibrium has to commit to its strategy at $t = 0$, which creates value *ex ante*. Specifically, a firm cannot condition its strategy on its competitor's future capital stock, which significantly mitigates value destruction arising from

future competition. Obviously, a monopoly industry has the highest value (given the same total industry capital stock $2k$) as it faces no competition.

As discussed earlier, the industry value, $V_{\text{closed}}^{\text{duo}}(x, k, k)$, increases with $\theta_1 + \theta_2$, which dictates the rate at which the industry invests: the larger this sum, the slower the industry invests.

Next, we compare a duopoly industry's investment rate to a monopoly's investment rate. Starting at $t = 0$ with no capital, $k_0^a = k_0^b = 0$, in a closed-loop equilibrium associated with a pair (θ_1, θ_2) , a duopoly industry's investment rate is proportional to a monopoly's investment rate with a coefficient of $\frac{4}{\theta_1 + \theta_2}$, which lies between $4/3$ and 2 (recall that Lemma 1 implies that $\theta_1 + \theta_2 \in [2, 3]$).

Duopolistic competition lowers the industry value at $t = 0$. As a duopoly's investment rate decreases with the sum $\theta_1 + \theta_2$, the industry value increases with this sum.¹⁴ The closer $\theta_1 + \theta_2$ is to the lower bound of 2 , the faster the industry invests and the lower the industry value. In the BP closed-loop equilibrium where $\theta_1 = \theta_2 = 1$, firms compete most fiercely. Indeed, the industry invests twice as fast as a monopoly: $\frac{dK_t^{\text{duo}}}{dK_t^{\text{mon}}} = 2$, thus completely eliminating the option value of waiting so that $V_{\text{BP}}^{\text{duo}}(x, 0, 0) = 0$. Next, we summarize these results.

Proposition 6 *As a fraction of the monopoly's investment rate, the duopoly industry investment rate in a closed-loop equilibrium is constant and decreases with $\theta_1 + \theta_2$.¹⁵*

$$\frac{dK_t^{\text{duo}}}{dK_t^{\text{mon}}} = \frac{4}{\theta_1 + \theta_2} \in \left(\frac{4}{3}, 2\right], \quad (56)$$

where $K_t^{\text{duo}} = K_t^a + K_t^b$. As a fraction of the monopoly value, the duopoly industry value in a closed-loop equilibrium is constant and increases with $\theta_1 + \theta_2$:

$$\frac{V_{\text{closed}}^{\text{duo}}(x, 0, 0)}{V^{\text{mon}}(x, 0)} = 4 \left[\frac{2}{\theta_1 + \theta_2} - \left(\frac{2}{\theta_1 + \theta_2} \right)^2 \right] < \frac{8}{9}. \quad (57)$$

Finally, we compare closed-loop equilibrium solutions with the open-loop equilibrium solution. In the open-loop equilibrium, a duopoly industry invests at a rate that is $1/3$ faster than a monopoly: $\frac{dK_t^{\text{duo}}}{dK_t^{\text{mon}}} = \frac{4}{3}$, which causes the industry value to be $1/9$ lower than that of a monopoly: $\frac{V_{\text{open}}^{\text{duo}}(x, 0, 0)}{V^{\text{mon}}(x, 0)} = \frac{8}{9}$. Consistent with our intuition, because a firm can respond to its competitor's investment strategy in a closed-loop equilibrium, but cannot in the open-loop equilibrium, the industry invests faster and hence its value is lower in all closed-loop equilibria

¹⁴The coefficient in (57), $4 \left[\frac{2}{\theta_1 + \theta_2} - \left(\frac{2}{\theta_1 + \theta_2} \right)^2 \right]$, increases with $\theta_1 + \theta_2$ as $\theta_1 + \theta_2 \in [2, 3]$.

¹⁵This is conditional on investing, i.e., $M_t > X_t > \rho(rp + c)$ and $dM_t > 0$ for $t > 0$.

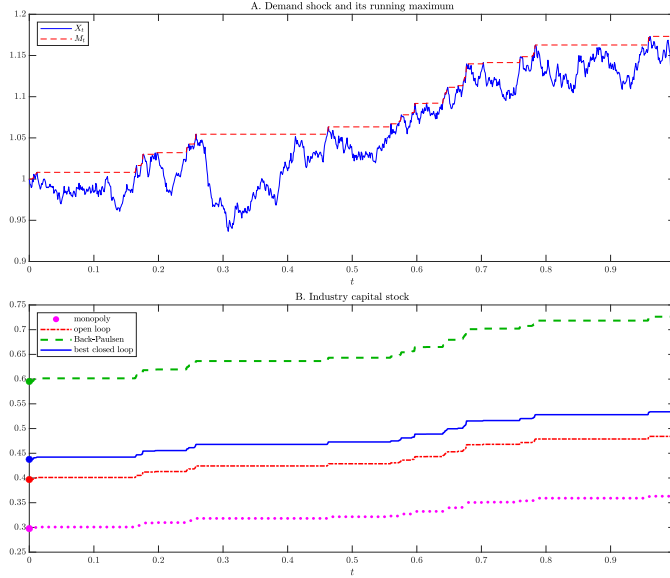


Figure 5: INDUSTRY INVESTMENT DYNAMICS. Panel A plots the running maximum M_t process (dashed red line) for a simulated demand shock path X_t (solid blue line). Panel B shows that at all t , the industry capital stock is larger under the best closed-loop equilibrium (solid blue line) than under the open-loop equilibrium (dash-dotted red line), attains the lowest level under monopoly (dotted magenta line) and the largest level in the BP equilibrium (dashed green line). Parameter values: $r = 0.06$, $\mu = 0.01$, $\sigma = 0.12$, $c = 0.1$, and $p = 1$.

than in the open-loop equilibrium.

7.2 A Quantitative Illustration

In this subsection, we show that a duopoly industry's investment dynamics differ substantially across different equilibria. In panel A of Figure 5, we simulate a demand shock $\{X_t\}$ path and report the associated running maximum $\{M_t\}$, which is noticeably smoother than $\{X_t\}$. Panel B compares the industry capital stock processes in three equilibria: the BP equilibrium, the best closed-loop equilibrium, and the open-loop equilibrium.

Among these three equilibria, in the BP equilibrium, a duopoly industry starts with the highest level of capital stock and increases its K_t^{duo} at the fastest rate (see the dashed green line in panel B of Figure 5). This is because firms compete most fiercely in this equilibrium.

In the open-loop equilibrium, a duopoly industry starts with the lowest level of capital stock and increases its capital stock at the slowest rate (see the dash-dotted red line in panel B). This is because a firm does not respond to its competitor's investment strategy at any $t > 0$, thus preserving maximal option value in a duopoly setting.

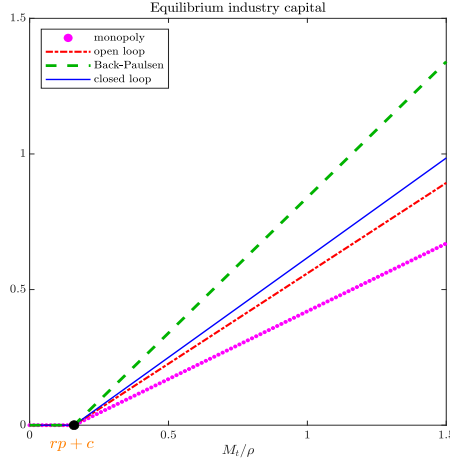


Figure 6: COMPARING EQUILIBRIUM INDUSTRY CAPITAL PROCESSES. We plot industry capital stock as a function of M_t/ρ . In all cases, industry investment starts when M_t/ρ exceeds $rp + c$ (the black dot) but responds to M_t/ρ at different rates. Parameter values: $r = 0.06$, $\mu = 0.01$, $\sigma = 0.12$, $c = 0.1$, and $p = 1$.

In the best closed-loop equilibrium, where the industry attains the highest value among all closed-loop equilibria, a duopoly industry starts with a level of capital stock that lies between those in the BP and open-loop equilibria and increases its capital stock at a rate that also lies between those in the BP and open-loop equilibria (see the solid blue line in panel B).

The intuition is as follows. On the one hand, firms preserve some option value of waiting by setting $\theta_1^* + \theta_2^* = 2.72$ in our numerical example, which is about 36% larger than $\theta_1 + \theta_2 = 2$ in the BP equilibrium. On the other hand, firms must choose strategies that are subgame perfect in the best closed-loop equilibrium. This important equilibrium restriction induces more aggressive competition than in the open-loop equilibrium, thereby eroding some of the option value. (Recall that $\theta_1 + \theta_2 = 3$ in the open-loop equilibrium, and that a larger sum implies slower industry investment and greater preservation of option value.)

Finally, a monopolist starts with a lower level of capital stock and invests less than in the open-loop equilibrium, as the absence of competition allows it to preserve the maximal option value.

Figure 6 illustrates industry dynamics as a function of M_t/ρ . In all duopoly equilibria, the industry (starting with no capital) remains dormant until M_t/ρ reaches $rp + c = 0.16$ in our numerical example (the black dot), and the industry then grows at different rates (slopes) in

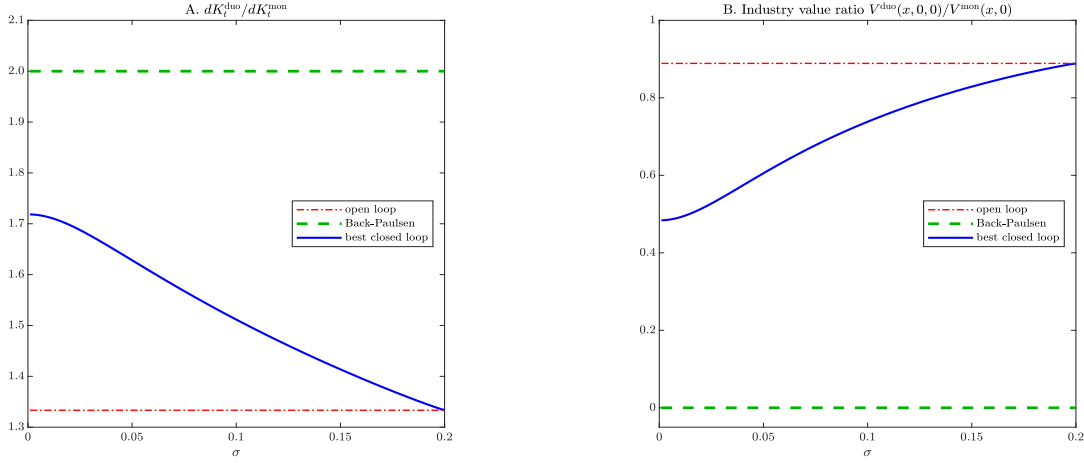


Figure 7: INDUSTRY INVESTMENT DYNAMICS AND VALUE. Panel A plots a duopoly industry investment as a fraction of a monopoly's investment and panel B plots the duopoly industry value as a fraction of the monopoly value for the open-loop equilibrium (dash-dotted red line), the BP closed-loop equilibrium (dashed green line), and the best closed-loop equilibrium (solid blue line). Parameter values: $r = 0.06$, $\mu = 0.01$, $c = 0.1$, and $p = 1$.

different equilibria.

A duopoly industry invests at the fastest rate in the BP equilibrium with a slope of 1 in our numerical example. The investment rate in the open-loop equilibrium is the slowest among all equilibria – it is only $2/3$ of the investment rate in the BP equilibrium. In the best closed-loop equilibrium, the industry invests at a rate of $2/(\theta_1^* + \theta_2^*) = 0.74$ in our numerical example, which is 26% lower than the investment rate in the BP equilibrium. That is, in the best closed-loop equilibrium, the industry invests much slower and attains a substantially higher value than in the BP equilibrium. Finally, a monopoly invests at a rate that is slower than that in any duopoly equilibria: it is only $1/2$ of the investment rate in the BP equilibrium.

Finally, in Figure 7 we show that the effects of volatility σ on investment dynamics (panel A) and industry value (panel B) are quantitatively large, even after we control for the standard option value of waiting for a monopoly. To control for the standard volatility effect, we report duopoly industry investment rates as a fraction of the monopoly's investment rate: In the BP closed-loop equilibrium $dK_t^{\text{duo}}/dK_t^{\text{mon}} = 2$ and in the open-loop equilibrium $dK_t^{\text{duo}}/dK_t^{\text{mon}} = 4/3$, both of which are independent of volatility σ . Figure 7 plots $dK_t^{\text{duo}}/dK_t^{\text{mon}}$ (panel A) and $V^{\text{duo}}(x, 0, 0)/V^{\text{mon}}(x, 0)$ (panel B) for the best closed-loop equilibrium together with these two special equilibria. As we increase volatility from around zero

toward 0.2, in our numerical example for the best closed-loop equilibrium, $dK_t^{\text{duo}}/dK_t^{\text{mon}}$ decreases by 23% from 1.72 to 1.33, causing the duopoly industry value as a fraction of the monopoly value, $V^{\text{duo}}(x, 0, 0)/V^{\text{mon}}(x, 0)$, to increase by 85% from 0.48 to 0.89. This monotonicity is consistent with the intuition that the industry as a whole competes less aggressively when demand uncertainty is higher as the industry's option value of waiting is larger.

8 Closed-loop Equilibria with Nonlinear Boundaries

So far, we have analyzed closed-loop equilibria with piecewise linear investment thresholds. In this section, we show that there also exist closed-loop equilibria with nonlinear investment thresholds. Unlike in a closed-loop equilibrium with a piecewise linear investment threshold, firm i 's marginal q equals the marginal cost of purchasing capital, p , in that

$$\left. \frac{\partial V^i(x, k^a, k^b)}{\partial k^i} \right|_{x=\mathcal{X}(k^a, k^b)} = p \quad (58)$$

holds not only on the segment of the investment threshold $\mathcal{X}(k^a, k^b)$ where $k^i < k^{-i}$, but also on the other segment of the investment threshold $\mathcal{X}(k^a, k^b)$ where $k^i > k^{-i}$.

We conjecture and verify that the nonlinear investment boundary takes the form:

$$\mathcal{X}(k^a, k^b) := \begin{cases} \rho(\mathcal{G}(k^a, k^b) + rp + c), & \text{if } k^a \leq k^b, \\ \rho(\mathcal{G}(k^b, k^a) + rp + c), & \text{if } k^a \geq k^b, \end{cases} \quad (59)$$

where $\mathcal{G}(k^a, k^b)$ is a nonlinear function to be determined. Internet Appendix IA.B provides Condition IA.1 under which the following four results hold in a closed-loop equilibrium with nonlinear thresholds: (i) a firm's marginal q cannot exceed the capital purchase cost p , as given in (26); (ii) firms earn positive profits so that (27) holds; (iii) a firm's cross marginal q is non-positive so that (54) holds; and (iv) a firm's marginal q equals p on the investment boundary $\mathcal{X}(k^a, k^b)$, as given in (58). Condition IA.1 places the boundary strictly between the BP and open-loop boundaries, requires $\mathcal{G}$ to be continuously differentiable and increasing in both capital stocks, and restricts its functional form through an implicit equation.

Theorem 4 *Under Condition IA.1, with $\mathcal{X}(k^a, k^b)$ given in (59), a strategy pair $(\varphi_*^a, \varphi_*^b) = \{(\mathbf{u}_*^a, \mathbf{w}_*^a), (\mathbf{u}_*^b, \mathbf{w}_*^b)\} \in \mathcal{S}$ constitutes a closed-loop equilibrium if $\min(k_0^a, k_0^b) \geq \underline{k}$, where $\underline{k} > 0$*

is the constant in Condition IA.1, and

$$\mathbf{u}_*^i(z) = \left[\frac{\mathbf{1}_{k^i < k^{-i}}}{\vartheta_2(k^a, k^b)} + \frac{\mathbf{1}_{k^i = k^{-i}}}{\vartheta_1(k^a, k^b) + \vartheta_2(k^a, k^b)} \right] \mathbf{1}_{x \geq \mathcal{X}(k^a, k^b)}, \quad (60)$$

$$\mathbf{w}_*^i(z) = \inf \left\{ \delta \geq 0 : \mathcal{X}(k^i + \delta, \max\{k^{-i}, \mathcal{K}(x)\}) \geq x \right\}, \quad (61)$$

for any $z = (x, k^a, k^b) \in \mathbb{Z} := (0, \infty) \times [\underline{k}, \infty) \times [\underline{k}, \infty)$. In (60)-(61),

$$\mathcal{K}(x) := \inf \{k \geq \underline{k} : \mathcal{X}(k, k) \geq x\} \quad (62)$$

and¹⁶

$$\vartheta_1(k^a, k^b) := \begin{cases} \frac{\partial \mathcal{X}(k^a, k^b)}{\partial k^b}, & \text{if } k^a \leq k^b, \\ \frac{\partial \mathcal{X}(k^a, k^b)}{\partial k^a}, & \text{if } k^a \geq k^b, \end{cases} \quad \text{and} \quad \vartheta_2(k^a, k^b) := \begin{cases} \frac{\partial \mathcal{X}(k^a, k^b)}{\partial k^a}, & \text{if } k^a \leq k^b, \\ \frac{\partial \mathcal{X}(k^a, k^b)}{\partial k^b}, & \text{if } k^a \geq k^b. \end{cases} \quad (63)$$

There exist many closed-loop equilibria with nonlinear boundaries. Below we characterize closed-loop equilibria in closed form for two specifications.

Proposition 7 *For any $\lambda > 0$, there exist closed-loop equilibria characterized by the following nonlinear investment boundaries $\mathcal{X}(k^a, k^b)$:*

(i) *If $\min(k_0^a, k_0^b) > \sqrt{\lambda}$, then*

$$\mathcal{X}(k^a, k^b) = \rho \left(k^a + k^b + \frac{\lambda}{\max\{k^a, k^b\}} + rp + c \right). \quad (64)$$

(ii) *If $\min(k_0^a, k_0^b) > \frac{3}{\lambda}$, then*

$$\mathcal{X}(k^a, k^b) = \rho \left((k^a + k^b) \frac{\lambda \max\{k^a, k^b\}}{\lambda \max\{k^a, k^b\} - 1} + rp + c \right). \quad (65)$$

Nonlinear triggers in (64) and (65) are captured by the terms involving the max operators. To illustrate industry capital dynamics in a closed-loop equilibrium, Figure 8 plots $K_t^{\text{duo}}/K_t^{\text{mon}}$ as a function of the running maximum M_t , using the nonlinear trigger in (64) with $\lambda = 1$.

Panel A shows that the industry investment dynamics are highly nonlinear, unlike the dynamics in the closed-loop equilibrium with linear triggers analyzed in Section 6. With a nonlinear trigger, K_t^{duo} is larger than in the best closed-loop equilibrium (with a linear trigger).

Panel B shows that the resulting industry value is lower in an equilibrium with a nonlinear trigger than in the best closed-loop equilibrium. The quantitative effect in our numerical example is large. For example, when $M_t = 5$, the duopoly industry capital stock is 62% larger

¹⁶At $k^a = k^b = k$, the derivatives in (63) are understood as the continuous extensions of the derivatives from the respective regions $k^a < k^b$ and $k^a > k^b$.

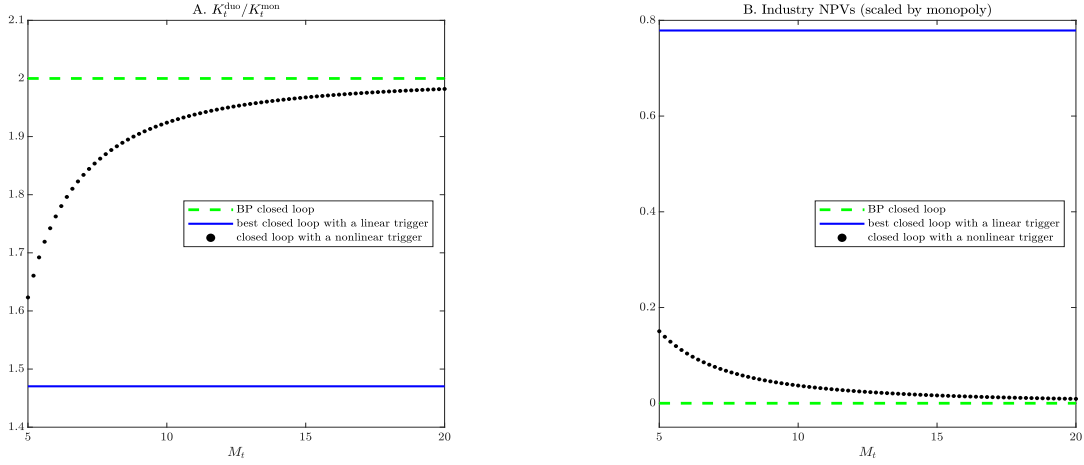


Figure 8: EQUILIBRIUM INDUSTRY CAPITAL PROCESSES AND INDUSTRY NET PRESENT VALUES (NPVs). Panel A plots $K_t^{\text{duo}}/K_t^{\text{mon}}$ as a function of the demand shock's running maximum M_t in a closed-loop equilibrium with a nonlinear investment trigger, the BP equilibrium, and the best closed-loop equilibrium with a linear investment trigger. Panel B plots the duopoly industry's NPV (scaled by a monopoly's NPV) in the respective equilibria, where a duopoly industry's NPV is defined as $V^{\text{duo}}(M_t, K_t^{\text{duo}}/2, K_t^{\text{duo}}/2) - pK_t^{\text{duo}}$. Parameter values: $r = 0.06$, $\mu = 0.01$, $\sigma = 0.12$, $c = 0.1$, and $p = 1$.

than a monopoly's: $K_t^{\text{duo}}/K_t^{\text{mon}} = 1.62$, reducing the industry's net present value (NPV) to only 15% of a monopoly's NPV. By contrast, the industry's NPV in the best closed-loop equilibrium is 78% of a monopoly's when $M_t = 5$. That is, the industry's NPV in the equilibrium with a nonlinear trigger is only 19% of that in the best closed-loop equilibrium (with a linear trigger) analyzed in the preceding section. This large value destruction occurs because firms compete much more aggressively in the equilibrium with a nonlinear trigger than in the best closed-loop equilibrium, significantly eroding the option value of waiting.

In the limit as $M_t \rightarrow \infty$, the nonlinear term in (64) becomes increasingly small relative to the linear terms, and the ratio $K_t^{\text{duo}}/K_t^{\text{mon}}$ converges to two, approaching the BP closed-loop equilibrium solution. In summary, firms compete less aggressively when market demand is low but more fiercely as market demand increases in a closed-loop equilibrium with a nonlinear investment trigger.

9 Conclusion

Working with Grenadier (2002)'s classic real-option game setting, Back and Paulsen (2009) propose using the closed-loop equilibrium solution concept, which allows a firm to respond to its competitor's strategy over time, to characterize investment strategies. They further identify a closed-loop equilibrium in which firms compete fiercely, yielding a perfectly competitive equilibrium outcome. Finally, they pose two important open questions: (1) how to define a closed-loop equilibrium and (2) whether a closed-loop equilibrium, other than the one they identified, exists.

Responding to these open questions posed by Back and Paulsen, we first define a closed-loop equilibrium, identify infinitely many closed-loop equilibria in which firms earn positive profits, and then derive the corresponding value functions and investment policies in closed form.

We also contribute to the q theory of investment literature (Lucas and Prescott, 1971; Hayashi, 1982; Abel and Eberly, 1994, 1996) by extending it to an imperfectly competitive equilibrium setting. When analyzing the duopolistic equilibrium, we focus on the best closed-loop equilibrium, the one associated with the highest firm value among all closed-loop equilibria. Consistent with our intuition, compared with the BP closed-loop equilibrium in which firms earn zero profits, the option value of waiting in the best closed-loop equilibrium is quantitatively large.

Bloom (2009) and the subsequent literature demonstrate that uncertainty shocks amplify the option value of waiting relative to standard real-options models. Extending our duopolistic setting to incorporate uncertainty shocks is a natural direction for future research; we conjecture that doing so would generate an even larger option value of waiting than implied by our baseline duopolistic q -theory model of investment.

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Appendices

A Sequential-game Formulation of the BP Equilibrium

In this appendix, we provide a sequential-game formulation of the BP closed-loop equilibrium. We decompose the leader's and follower's investment rates, dK_t^L and dK_t^F , over a small time interval dt into two components:

$$dK_t^L = dK_t^{L,C} + \Delta K_t^L, \quad (\text{A.1})$$

$$dK_t^F = dK_t^{F,C} + \Delta K_t^F, \quad (\text{A.2})$$

where $dK_t^{L,C}$ and $dK_t^{F,C}$ denote the continuous components of dK_t^L and dK_t^F , respectively, and $\Delta K_t^L := K_{t+}^L - K_t^L$ and $\Delta K_t^F := K_{t+}^F - K_t^F$ denote the instantaneous increases (jumps) of K^L and K^F , respectively. Assume that the jump and continuous components of the leader's investment strategy are given by

$$\Delta K_t^L = \mathbf{w}^L(Z_t), \quad (\text{A.3})$$

$$dK_t^{L,C} = u_t^L dM_t = \mathbf{u}^L(Z_t) dM_t, \quad (\text{A.4})$$

where $\mathbf{w}^L(\cdot) \geq 0$ and $\mathbf{u}^L(\cdot) \geq 0$ are functions of Z_t .

By contrast, the jump component of the follower's investment strategy, ΔK_t^F , is allowed to depend not only on the state, $Z_t = (X_t, K_t^a, K_t^b)$, but also on the jump component of the leader's strategy, ΔK_t^L , so that we can express ΔK_t^F as:

$$\Delta K_t^F = \mathbf{w}^F(Z_t, \Delta K_t^L), \quad (\text{A.5})$$

where $\mathbf{w}^F(\cdot, \cdot) \geq 0$ is a function of Z_t and ΔK_t^L . Similarly, we can express the continuous component of the follower's investment strategy $dK_t^{F,C}$ as:

$$dK_t^{F,C} = \mathbf{u}^F(Z_t, u_t^L) dM_t, \quad (\text{A.6})$$

where $\mathbf{u}^F(\cdot, \cdot) \geq 0$ is a function of Z_t and u_t^L .

We characterize the leader's and follower's investment strategies using their closed-loop strategies: $\varphi^i = (\mathbf{u}^i, \mathbf{w}^i)$ for $i = L$ and F . Feasible closed-loop strategies and closed-loop equilibria are defined as in Definitions 2 and 3, respectively.

Next, we summarize the BP closed-loop equilibrium solution (see Internet Appendix IA.E

for a proof.)

Theorem 5 *A strategy pair $(\varphi_*^L, \varphi_*^F) = \{(\mathbf{u}_*^L, \mathbf{w}_*^L), (\mathbf{u}_*^F, \mathbf{w}_*^F)\}$ constitutes a closed-loop equilibrium if*

$$\mathbf{u}_*^L(z) = \frac{1}{\rho} \mathbf{1}_{x \geq \mathcal{X}^{\text{BP}}(k^a, k^b)}, \quad (\text{A.7})$$

$$\mathbf{u}_*^F(z, u^L) = \left(\frac{1}{\rho} - u^L \right)^+ \mathbf{1}_{x \geq \mathcal{X}^{\text{BP}}(k^a, k^b)}, \quad (\text{A.8})$$

$$\mathbf{w}_*^L(z) = \left(\frac{x}{\rho} - (rp + c) - k^a - k^b \right)^+, \quad (\text{A.9})$$

$$\mathbf{w}_*^F(z, w^L) = \left(\frac{x}{\rho} - (rp + c) - k^a - k^b - w^L \right)^+, \quad (\text{A.10})$$

for all $z = (x, k^a, k^b) \in \mathbb{Z}$.

B Proofs

B.1 Regularity Conditions and Some Useful Technical Results

In this appendix, we first state regularity conditions and then establish some useful technical results under these regularity conditions (see Proposition 8.) Motivated by (31)–(33), we define the investment and inaction regions associated with the value functions, $V^a(z)$ and $V^b(z)$, as follows:

$$\mathbb{I}_i := \left\{ z \in \mathbb{Z} : \frac{\partial V^i(z)}{\partial k^i} = p \text{ and } \frac{\partial V^{-i}(z)}{\partial k^i} = 0 \right\}, \quad \mathbb{N}_i := \mathbb{Z} \setminus \mathbb{I}_i. \quad (\text{B.1})$$

Let $\mathbb{N}_{ab}$ denote the intersection of $\mathbb{N}_a$ and $\mathbb{N}_b$:

$$\mathbb{N}_{ab} := \mathbb{N}_a \cap \mathbb{N}_b. \quad (\text{B.2})$$

Let $\overline{\mathbb{N}}_{ab}$ denote the closure of $\mathbb{N}_{ab}$, which includes both its interior and boundary.

Condition A.1 *Assume $V^i \in \mathcal{C}^{2,1,1}(\overline{\mathbb{N}}_{ab}) \cap \mathcal{C}^{1,1,1}(\mathbb{Z})$ and equation (34) holds for any $z \in \mathbb{N}_{ab}$.*

Condition A.2 *For any initial state $(X_0, K_0^a, K_0^b) = (x, k^a, k^b) \in \mathbb{Z}$, the capital stock process pair (K^a, K^b) generated by $(\varphi_*^a, \varphi_*^b)$ has the following properties:*

- *At the initial time $t = 0$, the value-matching condition holds (from $t = 0$ to $t = 0+$):*

$$V^i(x, K_{0+}^a, K_{0+}^b) - p(K_{0+}^i - k^i) = V^i(x, k^a, k^b); \quad (\text{B.3})$$

- For all $t > 0$, $(X_t, K_t^a, K_t^b) \in \bar{\mathbb{N}}_{ab}$ and

$$K_t^i = K_{0+}^i + \int_{0+}^t \mathbf{1}_{\{Z_s \in \mathbb{I}_i\}} dK_s^i \quad (\text{B.4})$$

is continuous in $t > 0$, where the integrand is an indicator function.

- Regularity conditions hold: for any $z \in \mathbb{Z}$, $\lim_{n \rightarrow \infty} \mathbb{E}^z [e^{-rt_n} V^i(X_{t_n}, K_{t_n}^a, K_{t_n}^b)] = 0$ for some sequence $\{t_n\} \rightarrow \infty$ and $\mathbb{E}^z \left[\int_0^t \left| e^{-rs} \frac{\partial V^i}{\partial x}(X_s, K_s^a, K_s^b) \sigma X_s \right|^2 ds \right] < \infty$, for all $t \geq 0$.

Equation (B.3) is implied by the first-order condition for investment when firm i undertakes an initial lumpy investment. Using (B.4), we can show $dK_s^i = 0$ for $Z_s \notin \mathbb{I}_i$.

Proposition 8 *If a pair of functions, $(V^a(z), V^b(z))$, and a pair of strategies, $(\varphi_*^a, \varphi_*^b) \in \mathcal{S}$, satisfy Conditions A.1-A.2, then $V^a(z) = J^a(z; \varphi_*^a, \varphi_*^b)$ and $V^b(z) = J^b(z; \varphi_*^a, \varphi_*^b)$.*

The proof of Proposition 8 is provided in Internet Appendix IA.E.

B.2 Proofs of Theorem 3 and Propositions 3-4

In Internet Appendix IA.E, we will show that the BP solution constitutes a closed-loop equilibrium. Therefore, we only need to consider the case $(\theta_1, \theta_2) \in \Theta \setminus \{(1, 1)\}$ and verify the conditions stated in Theorem 2 and Proposition 8.

We first present an auxiliary lemma used to verify that inequalities (26) and (54), as well as Condition A.1, hold.

Lemma 2 *Under Condition 1, the function $V^i(z)$ in Proposition 3 satisfies inequalities (26) and (54), as well as Condition A.1.*

The proof of Lemma 2 is relegated to Internet Appendix IA.E.

Next, we verify that $(\varphi_*^a, \varphi_*^b) \in \mathcal{S}$ and that the functions $V^a(z)$ and $V^b(z)$ from Proposition 3, with the trigger function (43), satisfy Condition A.2. We organize this verification into three steps.

Step 1: Derive industry dynamics.

We first consider the region to the northeast of the ACE boundary in Figure 4, where $X_t \leq \mathcal{X}(K_t^a, K_t^b)$. Since $\mathcal{X}(k^a, k^b)$ increases in k^a and k^b , it follows from (49) that $\mathbf{w}_*^i(x, k^a, k^b) = 0$

for all $x \leq \mathcal{X}(k^a, k^b)$. Thus,

$$\Delta K_t^i = \mathbf{w}_*^i(X_t, K_t^a, K_t^b) = 0 \quad \text{for all } X_t \leq \mathcal{X}(K_t^a, K_t^b). \quad (\text{B.5})$$

Then, no lumpy investment occurs whenever $X_t \leq \mathcal{X}(K_t^a, K_t^b)$.

Second, we consider the OBCD square region in Figure 4. From (49) and $\mathcal{X}(\mathcal{K}(x), \mathcal{K}(x)) = x$, we have¹⁷

$$\Delta K_t^i = \mathcal{K}(X_t) - K_t^i > 0 \quad \text{when } (K_t^a, K_t^b) \in [0, \mathcal{K}(X_t)) \times [0, \mathcal{K}(X_t)), \quad (\text{B.6})$$

indicating that both firms invest lumpily to reach $(\mathcal{K}(X_t), \mathcal{K}(X_t))$.

Third, we consider the triangular region ABC in Figure 4. From (49) and the monotonicity of $\mathcal{X}(k^a, k^b)$ in k^a and k^b , we infer that

$$\Delta K_t^a = \inf\{\delta \geq 0 : \mathcal{X}(K_t^a + \delta, K_t^b) \geq X_t\} > 0 \text{ and } \Delta K_t^b = 0 \quad (\text{B.7})$$

$$\text{when } X_t > \mathcal{X}(K_t^a, K_t^b) \text{ and } K_t^b \geq \mathcal{K}(X_t), \quad (\text{B.8})$$

where we have used $K_t^a < \mathcal{K}(X_t)$. This indicates that firm b does not make a lumpy investment while firm a does at time t , moving instantaneously to $\Gamma_a(X_t)$. By symmetry, the jump component of firm i 's investment strategy in the triangular region CDE in Figure 4 can be derived analogously.

In summary, (B.5)-(B.8) imply the jump component of firm i 's investment strategy is uniquely determined. Next, we solve the continuous component of firm i 's investment strategy.

In equation (48), the indicator outside the square brackets implies that $\mathbf{u}_*^a(z) = \mathbf{u}_*^b(z) = 0$ in the inaction region where $x < \mathcal{X}(k^a, k^b)$. In the action region $x \geq \mathcal{X}(k^a, k^b)$ with $k^i < k^{-i}$, (48) implies $\mathbf{u}_*^i(z) = \frac{1}{\rho\theta_2}$, $\mathbf{u}_*^{-i}(z) = 0$. That is, only the smaller firm invests:

$$dK_t^i = \frac{dM_t}{\rho\theta_2} \text{ and } dK_t^{-i} = 0 \quad \text{when } K_t^i < K_t^{-i} \text{ and } \mathcal{X}(K_t^i, K_t^{-i}) = X_t. \quad (\text{B.9})$$

Combining the above with (B.7), we conclude that starting from within the ABC and CDE triangular regions, the smaller firm i increases its capital to the boundary $\Gamma_i(X_t)$, but the larger firm $-i$ chooses to be inactive. Subsequently, the smaller firm i invests continuously at a rate proportional to dM_t , as described in (B.9), and the larger firm $-i$ does not invest until the capital levels of both firms become equal.

¹⁷For $k^a < \mathcal{K}(x)$ and $k^b < \mathcal{K}(x)$, (49) implies $\mathcal{X}(k^i + \mathbf{w}_*^i(z), \mathcal{K}(x)) = x$. Thus, we must have $\mathbf{w}_*^i(z) = \mathcal{K}(x) - k^i$, $i = a, b$.

In the action region $x \geq \mathcal{X}(k^a, k^b)$ with $k^a = k^b$, (48) implies that

$$dK_t^a = dK_t^b = \frac{dM_t}{\rho(\theta_1 + \theta_2)} \quad \text{when } K_t^a = K_t^b \text{ and } \mathcal{X}(K_t^i, K_t^{-i}) = X_t. \quad (\text{B.10})$$

Combining the above with (B.6), we conclude that starting from within the OBCD square region, both firms invest so that $K_t^a = K_t^b = \mathcal{K}(M_t)$, reaching point C in Figure 4. Subsequently, the two firms invest at the same rate at all t and thus are always of the same size. In summary, (B.9)-(B.10) imply the continuous component of firm i 's investment strategy is uniquely determined.

Combining (B.5)-(B.10), we can obtain the industry dynamics:

$$K_t^i = \max \left\{ \min \left\{ \widehat{K}_t^i, \mathcal{K}_t^M \right\}, k_0^i \right\}, \quad \text{for } i = a, b, \quad (\text{B.11})$$

where $\widehat{K}_t^a$ and $\widehat{K}_t^b$, uniquely solve $\mathcal{X}(\widehat{K}_t^a, k_0^b) = M_t$ and $\mathcal{X}(k_0^a, \widehat{K}_t^b) = M_t$, respectively, and $\mathcal{K}_t^M = \mathcal{K}(M_t) = \frac{1}{\theta_1 + \theta_2} \left[\frac{M_t}{\rho} - (rp + c) \right]$.

Step 2: Verify $(\varphi_^a, \varphi_*^b) \in \mathcal{S}$.*

According to the results proved in Step 1, we conclude that the strategy $(\varphi_*^a, \varphi_*^b)$ satisfies the conditions in Definition 2-(i). Next, we proceed to verify the conditions in Definition 2-(ii).

By (B.11), we have $K_t^i \leq k_0^i \vee \mathcal{K}(M_t)$, $t \geq 0$. Since $\mathcal{K}(x) = \frac{1}{\theta_1 + \theta_2} \left[\frac{x}{\rho} - (rp + c) \right]$ is linear, there exists a constant $C_1 > 0$ such that $K_t^i \in [k_0^i, C_1(1 + M_t)]$, $\forall t \geq 0$, $i = a, b$. Then, for any finite $T > 0$ and $n > 1$, a sufficient condition for $\mathbb{E}^z [|K_T^i|^n] < \infty$ is

$$\mathbb{E}^z [|M_T|^n] < \infty. \quad (\text{B.12})$$

Since $\pi^i(x, k^a, k^b) = (x - (k^a + k^b) - c)k^i$, and since $K_t^i \in [k_0^i, C_1(1 + M_t)]$ and $X_t \in [0, M_t]$, $\forall t \geq 0$, there exists a constant $C_2 > 0$ such that $|\pi^i(X_t, K_t^a, K_t^b)| \leq C_2(1 + |M_t|^2)$, $\forall t \geq 0$.

Thus, a sufficient condition for (22) is

$$\int_0^{+\infty} \mathbb{E}^z [e^{-rt} |M_t|^2] dt < +\infty. \quad (\text{B.13})$$

By (2.16) in Chapter 4 of Friedman (2012), (B.12) holds for any $T > 0$ and $n > 1$. For any initial value z , there exists a constant $C_3 > 0$ such that

$$\begin{aligned} \mathbb{E}^z [M_t^2] &\leq 3X_0^2 + 3\mathbb{E}^z \left[\left(\int_0^t |\mu| X_s ds \right)^2 \right] + 3\mathbb{E}^z \left[\sup_{\tau \in [0, t]} \left(\int_0^\tau \sigma X_s d\mathcal{B}_s \right)^2 \right] \\ &\leq 3X_0^2 + (3\mu^2 t + C_3 \sigma^2) x_0^2 \int_0^t e^{(\sigma^2 + 2\mu)s} ds, \end{aligned} \quad (\text{B.14})$$

where the first inequality uses $X_t = X_0 + \int_0^t \mu X_s ds + \int_0^t \sigma X_s d\mathcal{B}_s$ and $|x_1 + x_2 + x_3|^2 \leq$

$3(|x_1|^2 + |x_2|^2 + |x_3|^2)$, and the second inequality follows from the Cauchy inequality and the Burkholder-Davis-Gundy Inequality. Since $\beta > 2$ implies $r > \sigma^2 + 2\mu$, we conclude from (B.14) that (B.13) holds. Therefore, the conditions in Definition 2-(ii) are satisfied, and $(\varphi_*^a, \varphi_*^b) \in \mathcal{S}$.

Step 3: Verify Condition A.2.

By (B.11), the definition of $V^i(z)$ in Proposition 3, and symmetry, it follows that (B.3) holds. By (B.11), we have $(X_t, K_t^a, K_t^b) \in \overline{\mathbf{N}}_{ab}$ for all $t > 0$, and K_t^i is continuous in $t > 0$ and satisfies (B.4).

It remains to verify the regularity conditions in Condition A.2. By symmetry, we focus on firm a . Denote

$$\begin{aligned}\mathcal{H}(k) &:= |H^a(k, k)| + k \left(\frac{3k + pr + c}{r} + \frac{\mathcal{X}(k, k)}{r - \mu} \right) \mathcal{X}(0, k)^{-\beta}, \\ \tilde{\mathcal{H}}(k) &:= |H^a(k, k)| + \frac{k^2}{r} \mathcal{X}(k, 0)^{-\beta},\end{aligned}$$

where $H^a(k^a, k^b)$ is given by (41)-(42). We first show that $\lim_{k \rightarrow +\infty} \mathcal{H}(k) = 0$, $\lim_{k \rightarrow +\infty} \tilde{\mathcal{H}}(k) = 0$, and

$$|H^a(k^a, k^b)| \leq \mathcal{H}(k^b) \mathbf{1}_{k^a \leq k^b} + \tilde{\mathcal{H}}(k^a) \mathbf{1}_{k^a > k^b}, \quad \forall (k^a, k^b) \in \mathbb{R}_+^2. \quad (\text{B.15})$$

Since $\lim_{k \rightarrow +\infty} H^a(k, k) = 0$ and $\beta > 2$, and $\mathcal{X}(k, k)$ and $\mathcal{X}(0, k)$ are positive and linearly increasing in k , we have $\lim_{k \rightarrow +\infty} \mathcal{H}(k) = 0$. By (42), for $k^b \geq k^a \geq 0$, we have

$$|H^a(k^a, k^b)| = \left| H^a(k^b, k^b) - \int_{k^a}^{k^b} \left(\frac{2\alpha + k^b + rp + c}{r} - \frac{\mathcal{X}(\alpha, k^b)}{r - \mu} \right) \mathcal{X}(\alpha, k^b)^{-\beta} d\alpha \right| \leq \mathcal{H}(k^b),$$

where the inequality follows from the monotonicity of $\mathcal{X}(k^a, k^b)$ in $k^a \in [0, k^b]$. Similarly, using (42), we can obtain $|H^a(k^a, k^b)| = \left| H^a(k^a, k^a) - \int_{k^b}^{k^a} \frac{k^a}{r} \mathcal{X}(k^a, \alpha)^{-\beta} d\alpha \right| \leq \tilde{\mathcal{H}}(k^a)$ for $k^a \geq k^b \geq 0$. Thus, we can see (B.15) holds.

Fix any $(k^a, k^b) \in \overline{\mathbf{N}}_{ab}(x)$, i.e., $x \leq \mathcal{X}(k^a, k^b)$. If $k^a \leq k^b$, then $\mathcal{X}(k^b, k^b) \geq \mathcal{X}(k^a, k^b) \geq x$, which implies that $k^b \geq \mathcal{K}(x)$. If $k^a \geq k^b$, then $\mathcal{X}(k^a, k^a) \geq \mathcal{X}(k^a, k^b) \geq x$, which implies that $k^a \geq \mathcal{K}(x)$. Hence, we have

$$k^b \mathbf{1}_{k^a \leq k^b} + k^a \mathbf{1}_{k^a > k^b} \geq \mathcal{K}(x). \quad (\text{B.16})$$

Since $\lim_{k \rightarrow \infty} \mathcal{H}(k) = 0$, $\lim_{k \rightarrow \infty} \tilde{\mathcal{H}}(k) = 0$, and $\lim_{x \rightarrow +\infty} \mathcal{K}(x) = \infty$, for any $\delta > 0$, there exists $x_\delta > 0$ such that $|\mathcal{H}(k^b)| \leq \delta$ and $|\tilde{\mathcal{H}}(k^a)| \leq \delta$ for any $k^a \geq \mathcal{K}(x_\delta)$ and $k^b \geq \mathcal{K}(x_\delta)$. Combining (B.15) and (B.16), we see that $k^b \mathbf{1}_{k^a \leq k^b} + k^a \mathbf{1}_{k^a > k^b} \geq \mathcal{K}(x_\delta)$ and $|H^a(k^a, k^b)| \leq \delta$

when $\mathcal{X}(k^a, k^b) \geq x_\delta$. Denote $\tau_\delta := \inf\{t \geq 0 : X_t \geq x_\delta\}$. Then for any $t > 0$,

$$|\mathbb{E}^z[e^{-rt}H^a(K_t^a, K_t^b)X_t^\beta \mathbf{1}_{t \geq \tau_\delta}]| \leq \delta |\mathbb{E}^z[e^{-rt}X_t^\beta]| = \delta x^\beta, \quad (\text{B.17})$$

where the inequality follows from $\mathcal{X}(K_t^a, K_t^b) \geq \mathcal{X}(K_{\tau_\delta}^a, K_{\tau_\delta}^b) \geq X_{\tau_\delta} = x_\delta$, $\forall t \geq \tau_\delta$ and $|H^a(k^a, k^b)| \leq \delta$ for $\mathcal{X}(k^a, k^b) \geq x_\delta$, and the equality holds as $e^{-rt}X_t^\beta$ is a martingale. The continuity of $H^a(k^a, k^b)$ in $(k^a, k^b) \in \mathbb{R}_+^2$ implies that $H^a(k^a, k^b)$ is bounded in the compact set $\{(k^a, k^b) \in \mathbb{R}_+^2 : \mathcal{X}(k^a, k^b) \leq x_\delta\}$. Owing to $|H^a(k^a, k^b)| \leq \delta$ for $\mathcal{X}(k^a, k^b) \geq x_\delta$, $H^a(k^a, k^b)$ is bounded in $(k^a, k^b) \in \mathbb{R}_+^2$. Recalling that $r > 0$ and $X_t \geq 0$, $\forall t \geq 0$, we infer that $\lim_{t \rightarrow +\infty} \mathbb{E}^z[e^{-rt}H^a(K_t^a, K_t^b)X_t^\beta \mathbf{1}_{t < \tau_\delta}] = 0$. Combining this with (B.17), we have $\lim_{t \rightarrow +\infty} \mathbb{E}^z[e^{-rt}H^a(K_t^a, K_t^b)X_t^\beta] \leq \delta x^\beta$. Since $\delta > 0$ is arbitrary, we obtain

$$\lim_{t \rightarrow \infty} \mathbb{E}^z[e^{-rt}H^a(K_t^a, K_t^b)X_t^\beta] = 0. \quad (\text{B.18})$$

According to (B.14), $r > 0$ and $r > \sigma^2 + 2\mu$, we have

$$\lim_{t \rightarrow \infty} \mathbb{E}^z[e^{-rt}M_t^2] = 0. \quad (\text{B.19})$$

We derive from (22) and Tonelli's Theorem that $\int_0^\infty \mathbb{E}^z[e^{-rt}|\pi^i(X_t, K_t^a, K_t^b)|]dt < \infty$. It follows that there exists a sequence $\{t_n\}$ of real numbers such that $\lim_{n \rightarrow \infty} t_n = \infty$ and

$$\lim_{n \rightarrow \infty} \mathbb{E}^z[e^{-rt_n}\pi^i(X_{t_n}, K_{t_n}^a, K_{t_n}^b)] = 0. \quad (\text{B.20})$$

Since $\pi^a(x, k^a, k^b) = k^a(x - \frac{1}{4}k^a) - k^a(\frac{3}{4}k^a + k^b + c) \leq k^a(x - \frac{1}{4}k^a)$ for any $(x, k^a, k^b) \in \mathbb{R}_+^3$, we derive from (B.20) that

$$\liminf_{n \rightarrow \infty} \mathbb{E}^z\left[e^{-rt_n}K_{t_n}^a\left(X_{t_n} - \frac{1}{4}K_{t_n}^a\right)\right] \geq 0. \quad (\text{B.21})$$

In addition, we have

$$0 \geq \limsup_{n \rightarrow \infty} \mathbb{E}^z\left[e^{-rt_n}\left(-X_{t_n}^2 + K_{t_n}^a\left(X_{t_n} - \frac{1}{4}K_{t_n}^a\right)\right)\right] = \limsup_{n \rightarrow \infty} \mathbb{E}^z\left[e^{-rt_n}K_{t_n}^a\left(X_{t_n} - \frac{1}{4}K_{t_n}^a\right)\right], \quad (\text{B.22})$$

where the first inequality follows from $-X_{t_n}^2 + K_{t_n}^a(X_{t_n} - \frac{1}{4}K_{t_n}^a) = -(X_{t_n} - \frac{1}{2}K_{t_n}^a)^2$, and the equality uses (B.19). Hence, we derive from (B.21) and (B.22) that $\lim_{n \rightarrow \infty} \mathbb{E}^z\left[e^{-rt_n}K_{t_n}^a\left(X_{t_n} - \frac{1}{4}K_{t_n}^a\right)\right] = 0$. Combining it, (B.20), and $\pi^a(x, k^a, k^b) = k^a(x - \frac{1}{4}k^a) - k^a(\frac{3}{4}k^a + k^b + c)$, we conclude that $\lim_{n \rightarrow \infty} \mathbb{E}^z[e^{-rt_n}K_{t_n}^a(\frac{3}{4}K_{t_n}^a + K_{t_n}^b + c)] = 0$.

Noticing $K^a \geq 0$, $K^b \geq 0$, and $c \geq 0$, we have

$$\lim_{n \rightarrow \infty} \mathbb{E}^z[e^{-rt_n}(K_{t_n}^a)^2] = 0, \quad \lim_{n \rightarrow \infty} \mathbb{E}^z[e^{-rt_n}K_{t_n}^a K_{t_n}^b] = 0. \quad (\text{B.23})$$

Combining (B.23) with (B.19) and using the Cauchy-Schwarz inequality, we obtain

$$\lim_{n \rightarrow \infty} \mathbb{E}^z[e^{-rt_n} K_{t_n}^a X_{t_n}] = 0. \quad (\text{B.24})$$

Since we have shown $X_t \leq \mathcal{X}(K_t^a, K_t^b)$ for any $t > 0$, it follows from (B.18) and (B.23)-(B.24) that $\lim_{n \rightarrow \infty} \mathbb{E}^z[e^{-rt_n} V^a(X_{t_n}, K_{t_n}^a, K_{t_n}^b)] = 0$, where $V^a(X_t, K_t^a, K_t^b)$ is given by (35).

Using (35), we have $V_x^a(x, k^a, k^b) = \frac{k^a}{r-\mu} + \beta H^a(k^a, k^b) x^{\beta-1}$ for any $(k^a, k^b) \in \bar{\mathbf{N}}_{ab}(x)$. Thus, there exists a constant $C_1 > 0$ such that for any $(k^a, k^b) \in \bar{\mathbf{N}}_{ab}(x)$,

$$\begin{aligned} |V_x^a(x, k^a, k^b) x|^2 &\leq 2 \left(\frac{k^a x}{r-\mu} \right)^2 + 2\beta^2 |H^a(k^a, k^b)|^2 x^{2\beta} \\ &\leq C_1 ((k^a)^4 + x^4 + x^{2\beta}) \leq C_1 ((k^a)^4 + 1 + 2x^{2\beta}), \end{aligned} \quad (\text{B.25})$$

where the second inequality follows from the boundedness of $H^a(k^a, k^b)$ in $\mathbb{R}_+^2$, and the last inequality uses $\beta > 2$ and $x^4 < 1 + x^{2\beta}$. Since $K_t^a \geq 0$ is nondecreasing in t , and since Step 2 establishes that $\mathbb{E}^z[|K_t^a|^n] < +\infty$ for any $t > 0$, $n > 1$, it follows that $\mathbb{E}^z[\int_0^t |K_s^a|^4 ds] \leq t \mathbb{E}^z[|K_t^a|^4] < +\infty$, $\forall t > 0$. Additionally, direct computation yields $\mathbb{E}^z[X_t^{2\beta}] = x^{2\beta} e^{\beta t(2\mu + (2\beta-1)\sigma^2)}$, implying $\mathbb{E}^z[\int_0^t X_s^{2\beta} ds] < \infty$, $\forall t > 0$. We then conclude from $X_t \leq \mathcal{X}(K_t^a, K_t^b)$, $t > 0$, and (B.25) that $\mathbb{E}^z \left[\int_0^t |e^{-rs} V_x^a(X_s, K_s^a, K_s^b) \sigma X_s|^2 ds \right] < +\infty$, $\forall t \geq 0$.

It remains to verify the value comparison (27). We first establish a general comparison result and then apply it to the piecewise linear investment boundary.

Lemma 3 *Let $V^i(z)$ denote firm i 's value associated with the symmetric investment threshold $\mathcal{X}(k^a, k^b)$. Suppose that $\frac{\partial V^i(x, k^a, k^b)}{\partial k^i} \leq p$, $\frac{\partial V^i(x, k^a, k^b)}{\partial k^{-i}} \leq 0$, and that $\mathcal{X}^{\text{open}}(k^a, k^b) \geq \mathcal{X}(k^a, k^b) > \mathcal{X}^{\text{BP}}(k^a, k^b)$ for any $x > 0$ and $(k^a, k^b) \in \mathbb{D}$, where $\mathcal{X}^{\text{open}}(k^a, k^b)$ is defined in (14), $\mathcal{X}^{\text{BP}}(k^a, k^b)$ is defined in (15) and $\mathbb{D} = \{(k^a, k^b) \in [0, \infty) \times [0, \infty) : k^i > l(k^{-i} - \underline{k}) + \underline{k}, i = a, b\}$ for some $\underline{k} \geq 0$ and $l \in [0, 1)$. Then $V^i(x, k^a, k^b) > V_{\text{BP}}^i(x, k^a, k^b)$ for all $x > 0$ and $(k^a, k^b) \in \bar{\mathbb{D}} = \{(k^a, k^b) \in [0, \infty) \times [0, \infty) : k^i \geq l(k^{-i} - \underline{k}) + \underline{k}, i = a, b\}$.*

Lemma 4 *Let $V^i(z)$ denote firm i 's value associated with the investment threshold $\mathcal{X}(k^a, k^b)$ given in (43), where $(\theta_1, \theta_2) \in \Theta \setminus \{(1, 1)\}$. Then, $V^i(z) > V_{\text{BP}}^i(z)$ for all $z \in \mathbb{Z}$.*

Internet Appendix IA.E provides proofs of Lemmas 3–4. The three steps above and Lemma 2 allow us to apply Proposition 8. Together with Lemma 4, Lemma 2 verifies the conditions of Theorem 2, establishing Theorem 3. The value formulas and the cross marginal q inequality yield Propositions 3 and 4, respectively.